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Research Article

Some congruences with $q-$binomial coefficients and $q-$harmonic numbers

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Abstract

In this paper, considering $q-$analogues and $q-$combinatorial identities, we gave some congruences including $q-$binomial coefficients and $q-$ harmonic numbers. For example, for any prime number p and $\alpha \in \mathbb{Z}^+$,

$$\begin{aligned}&\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk+\binom{k+1}{2}+k} [k]_q \begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \\ \equiv\ & \frac{q^{1-\alpha p}}{(1-q^2)^2} \left(q^{\alpha p+2} (q^p - 2) + q^{\alpha p} - q^p + q^2 \right) [p-1]_q \pmod{[p]_q^3}.\end{aligned}$$

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1. Introduction

Harmonic numbers H_n are those rational numbers defined by

$$H_0 = 0, \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \in \mathbb{Z}^+,$$

and for each $m = 2, 3, ...,$ harmonic numbers of order m are those rational numbers defined by

$$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m} \quad \text{for } n \in \mathbb{Z}^+.$$

In [11], Sun gave that for an odd prime number p and $1 \le k \le p-1$,

$$(-1)^k \binom{p-1}{k} \equiv 1 - pH_k + \frac{1}{2} p^2 \left(H_k^2 - H_{k,2} \right) \left(\bmod\ p^3 \right).$$

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In [2], Cai and Granville obtained that for any prime number $p \geq 5$,

$$\sum_{k=0}^{p-1} (-1)^{(a-1)k} \binom{p-1}{k}^a \equiv 2^{a(p-1)} \pmod{p^3},$$

and in [9], Pan showed that for an odd prime number p,

$$\sum_{k=0}^{p-1} (-1)^{(a-1)k} \binom{p-1}{k}^a \equiv 2^{a(p-1)} + \frac{a(a-1)(3a-4)}{48} p^3 B_{p-3} \pmod{p^4},$$

where a is a positive integer.

The $q-$harmonic numbers are given by

$$H_n(q) = \sum_{k=1}^{n} \frac{1}{[k]_q} \text{ and } \widetilde{H}_n(q) = \sum_{k=1}^{n} \frac{q^k}{[k]_q},$$

where $[0]_q = 1$ and $[k]_q = 1 + q + q^2 + ... + q^{k-1}$.

The $q-$harmonic numbers of order m are given by

$$\widetilde{H}_{n,m}(q) = \sum_{k=1}^{n} \frac{q^k}{[k]_q^m}.$$

It is clearly seen that for $p, k, \alpha \in \mathbb{Z}^+$ with $\alpha p - k > 0$,

$$[\alpha p - k]_q = [p]_q [\alpha]_{q^p} - q^{\alpha p - k} [k]_q . \tag{1.1}$$

A $q-$analogue of the binomial coefficient is the $q-$binomial coefficient, which is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! \, [n-k]_q!},$$

for $n, k \in \mathbb{N}$ with $n \geq k$, where

$$[n]_q! = [1]_q [2]_q \dots [n]_q$$

is the $q-$factorial. It is clear that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

where $\binom{n}{k}$ is the usual binomial coefficient.

In [7], Kızılateş and Tuğlu gave that for an integer number $n > 1$,

$$\sum_{k=1}^{n-1} q^k \widetilde{H}_k(q) = [n]_q \left(\widetilde{H}_n(q) - q \right), \tag{1.2}$$

and

$$\sum_{k=1}^{n-1} q^k [k]_q \widetilde{H}_k(q) = \frac{q}{1+q} [n]_q [n-1]_q \left(\widetilde{H}_n(q) - \frac{q^2}{1+q} \right). \tag{1.3}$$

In [8], Liu et al. investigated that for a positive odd integer n,

$$\begin{aligned} &\sum_{k=0}^{n-1} (-1)^{(a-1)k} q^{a\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q^a \\ \equiv \;& (-q;q)_{n-1}^a + \frac{a(a-1)(n^2-1)}{24} (1-q)^2 [n]_q^2 \pmod{\Phi_n(q)^3}, \end{aligned}$$

where a is a positive integer, $(x;q)_n$ is the $q-$Pochhammer symbol and $\Phi_n(q)$ is the nth cyclotomic polynomial.

We will refer to the book [3] for notation and basic facts on $q-$calculus. Also, in [4–6,10], the authors obtained some sums involving $q-$analogue. For example, in [4], Kim proved the formula for the $q-$analogue of $\sum_{j=1}^{k-1} j$ as follows: For $n \geq 1$,

$$\sum_{k=1}^{n} q^{k} [k]_q = \frac{q}{1+q} [n]_q [n+1]_q . \tag{1.4}$$

In [1], Abel's partial summation formula asserts that for every pair of families $(a_k)_{k=1}^{n}$ and $(b_k)_{k=1}^{n}$ of complex numbers, there is the relation

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} \left[(a_k - a_{k+1}) \left(\sum_{j=1}^{k} b_j \right) \right] + a_n \left(\sum_{j=1}^{n} b_j \right). \tag{1.5}$$

In this paper, considering $q-$analogues and $q-$combinatorial identities, we gave some congruences including $q-$binomial coefficients and $q-$harmonic numbers. For example, for any prime number p and $\alpha \in \mathbb{Z}^{+}$,

$$\begin{aligned} &\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk+\binom{k+1}{2}+k} [k]_q \begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \\ \equiv\ & \frac{q^{1-\alpha p}}{(1-q^2)^2} \left(q^{\alpha p+2} (q^p - 2) + q^{\alpha p} - q^p + q^2 \right) [p-1]_q \pmod{[p]_q^3}. \end{aligned}$$

2. Some results with congruences

In this section, firstly, we will start some auxiliary lemmas and then give main theorems:

Lemma 2.1. *Let p is any prime number and $\alpha \in \mathbb{Z}^{+}$. For $1 \leq k \leq \alpha p - 1$,*

$$\begin{aligned} (-1)^k q^{-\alpha pk+\binom{k+1}{2}} \begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \equiv\ & 1 - [p]_q [\alpha]_{q^p} q^{-\alpha p} \widetilde{H}_k(q) \\ &+ \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \Big(\widetilde{H}_k^2(q) - q^k \widetilde{H}_{k,2}(q) \\ &\quad - (1-q) \left([k]_q \widetilde{H}_{k,2}(q) - \widetilde{H}_k(q) \right) \Big) \left(\mathit{mod}\ [p]_q^3 \right). \end{aligned} \tag{2.1}$$

Proof. From (1.1), it is seen that

$$\begin{aligned} &\begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \\ =\ & \frac{[\alpha p - 1]_q [\alpha p - 2]_q \dots [\alpha p - k]_q}{[1]_q [2]_q \dots [k]_q} \\ =\ & \frac{\left([p]_q [\alpha]_{q^p} - q^{\alpha p - 1} [1]_q \right) \dots \left([p]_q [\alpha]_{q^p} - q^{\alpha p - k} [k]_q \right)}{[1]_q [2]_q \dots [k]_q} \\ \equiv\ & (-1)^k q^{-\alpha pk+\binom{k+1}{2}} \\ & \times \Big(1 - [p]_q [\alpha]_{q^p} q^{-\alpha p} \widetilde{H}_k(q) + \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \left(\widetilde{H}_k^2(q) - q^k \widetilde{H}_{k,2}(q) \right) \\ & \quad - \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} (1-q) \left([k]_q \widetilde{H}_{k,2}(q) - \widetilde{H}_k(q) \right) \Big) \quad \left(\mathrm{mod}\ [p]_q^3 \right), \end{aligned}$$

as claimed. ∎

Lemma 2.2. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n-1} q^k \widetilde{H}_{k,2}(q) = [n]_q \widetilde{H}_{n,2}(q) - \widetilde{H}_n(q). \tag{2.2}$$

Proof. From definition of $[n]_q$, the proof of (2.2) is clear. □

Lemma 2.3. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{q^{2k}}{[k]_q} = \widetilde{H}_n(q) - q(1-q)[n]_q, \tag{2.3}$$

and

$$\sum_{k=1}^{n} \frac{q^{2k}}{[k]_q^2} = q^n \widetilde{H}_{n,2}(q) + (1-q)\left([n]_q \widetilde{H}_{n,2}(q) - \widetilde{H}_n(q)\right) + \widetilde{H}_n^2(q). \tag{2.4}$$

Proof. Using (1.2) and (1.5), the proof of (2.3) is given and from (2.2), the proof of (2.4) is obtained. □

Lemma 2.4. *For $n > 1$, we have*

$$\sum_{k=1}^{n} \frac{q^k}{[k]_q} \widetilde{H}_k(q) = \frac{1}{2}\left(\widetilde{H}_{n,2}(q) + \widetilde{H}_n^2(q) - (1-q)\widetilde{H}_n(q)\right), \tag{2.5}$$

and

$$\begin{aligned}\sum_{k=1}^{n-1} q^k [k]_q \widetilde{H}_k^2(q) &= \frac{q}{1+q}\widetilde{H}_n(q)\left([n+1]_q [n]_q \widetilde{H}_n(q) - 2\frac{q^2}{q+1}[n]_q [n-1]_q + \frac{1}{q}\right) \\ &\quad + \frac{q}{2(1+q)}[n]_q\left(q[n]_q + q(q-1)\frac{q^n+1}{(q+1)^2} - \frac{2}{(q+1)}[3]_q\right).\end{aligned}$$

Proof. Firstly, we will give proof of (2.5). From (1.5), we get

$$\sum_{k=1}^{n} \frac{q^k}{[k]_q}\widetilde{H}_k(q) = \widetilde{H}_n^2(q) - \sum_{k=1}^{n-1}\frac{q^{k+1}}{[k+1]_q}\widetilde{H}_k(q) = \widetilde{H}_n^2(q) - \sum_{k=1}^{n}\frac{q^k}{[k]_q}\widetilde{H}_k(q) + \sum_{k=1}^{n}\frac{q^{2k}}{[k]_q^2},$$

and from (2.4), the desired result is obtained. Secondly, (1.5) yields that

$$\begin{aligned}&2\sum_{k=0}^{n} q^k [k]_q \widetilde{H}_k^2(q) \\ =\ & [n+1]_q^2 \widetilde{H}_{n+1}^2(q) - \sum_{k=0}^{n} q^{2k}\widetilde{H}_k^2(q) - 2q\sum_{k=1}^{n} q^k [k]_q \widetilde{H}_k(q) - 2q\sum_{k=0}^{n} q^{2k}\widetilde{H}_k(q) \\ & \qquad - \sum_{k=0}^{n} q^{2k+2} \\ =\ & [n+1]_q^2 \widetilde{H}_{n+1}^2(q) - q^n [n+1]_q \widetilde{H}_{n+1}^2(q) + q^n(1+q)[n+1]_q\left(\widetilde{H}_{n+1}(q) - q\right) \\ & + q^{2n+1}\widetilde{H}_{n+1}(q) - \frac{1-q}{q}\left\{\sum_{k=0}^{n} q^k [k]_q \widetilde{H}_k^2(q) - (1+q)\sum_{k=2}^{n} q^k [k]_q\left(\widetilde{H}_k(q) - q\right)\right\} \\ & + \frac{1-q}{q}\sum_{k=2}^{n} q^{2k}\widetilde{H}_k(q) - 2q\sum_{k=1}^{n} q^k [k]_q \widetilde{H}_k(q) - 2q\sum_{k=0}^{n} q^{2k}\widetilde{H}_k(q) - \sum_{k=0}^{n} q^{2k+2}.\end{aligned}$$

With the help of (1.5), it is seen that

$$\sum_{k=1}^{n} q^{2k}\widetilde{H}_k(q) = \frac{1-q}{q}\left(\sum_{k=1}^{n} q^k [k]_q \widetilde{H}_k(q) - q^2\right) \tag{2.6}$$

$$-\left(1-q\right)\sum_{k=1}^{n}q^{k}\left[k\right]_{q}+q\left(1-q\right)+q^{n}\left[n+1\right]_{q}\left(\widetilde{H}_{n+1}\left(q\right)-q\right).$$

By (1.3) and (2.6), the proof of the final equality is given. □

Lemma 2.5. *For $n>1$, we have*

$$\sum_{k=1}^{n-1}q^{2k}\widetilde{H}_{k,2}\left(q\right)=\frac{1}{1+q}\left(\left[2n\right]_{q}\widetilde{H}_{n,2}\left(q\right)-2\widetilde{H}_{n}\left(q\right)+q\left(1-q^{n}\right)\right),\tag{2.7}$$

and

$$\sum_{k=1}^{n-1}q^{k}\left[k\right]_{q}\widetilde{H}_{k,2}\left(q\right)=\frac{1}{1+q}\left(q\left[n\right]_{q}\left(\left[n-1\right]_{q}\widetilde{H}_{n,2}\left(q\right)-1\right)+\widetilde{H}_{n}\left(q\right)\right).\tag{2.8}$$

Proof. Observed that

$$\begin{aligned}
&\sum_{k=1}^{n}\left[k\right]_{q}^{2}\widetilde{H}_{k,2}\left(q\right)+\left[n+1\right]_{q}^{2}\widetilde{H}_{n+1,2}\left(q\right)\\
=\;&\sum_{k=1}^{n+1}\left[k\right]_{q}^{2}\widetilde{H}_{k,2}\left(q\right)=\sum_{k=0}^{n}\left[k+1\right]_{q}^{2}\widetilde{H}_{k+1,2}\left(q\right)\\
=\;&\sum_{k=0}^{n}\left[k\right]_{q}^{2}\widetilde{H}_{k,2}\left(q\right)+2\sum_{k=0}^{n}q^{k}\left[k\right]_{q}\widetilde{H}_{k,2}\left(q\right)+\sum_{k=0}^{n}q^{2k}\widetilde{H}_{k,2}\left(q\right)-q\left[n+1\right]_{q},
\end{aligned}$$

and thus

$$2\sum_{k=1}^{n}q^{k}\left[k\right]_{q}\widetilde{H}_{k,2}\left(q\right)=\left[n+1\right]_{q}^{2}\widetilde{H}_{n+1,2}\left(q\right)-\sum_{k=0}^{n}q^{2k}\widetilde{H}_{k,2}\left(q\right)-q\left[n+1\right]_{q}.\tag{2.9}$$

By (1.2) and (1.5), we write

$$\begin{aligned}
\sum_{k=1}^{n}q^{2k}\widetilde{H}_{k,2}\left(q\right)\;=\;&\frac{1-q}{q}\sum_{k=1}^{n}q^{k}\left[k\right]_{q}\widetilde{H}_{k,2}\left(q\right)\qquad(2.10)\\
&-\frac{1-q}{q}\left[n+1\right]_{q}\left(\widetilde{H}_{n+1}-q\right)+q^{n}\left[n+1\right]_{q}\widetilde{H}_{n+1,2}\left(q\right)-q^{n}\widetilde{H}_{n+1}\left(q\right).
\end{aligned}$$

From here, with the help of (2.9) and (2.10), the proofs of (2.7) and (2.8) are obtained. □

Lemma 2.6. *For $n\geq1$, we have*

$$\sum_{k=1}^{n-1}q^{k}\widetilde{H}_{k}^{2}\left(q\right)=\left[n\right]_{q}\widetilde{H}_{n}^{2}\left(q\right)-q^{n}\widetilde{H}_{n}\left(q\right)-\left(1+q\right)\left[n\right]_{q}\left(\widetilde{H}_{n}\left(q\right)-q\right),\tag{2.11}$$

and

$$\begin{aligned}
\sum_{k=1}^{n}q^{k}\widetilde{H}_{k}\left(q\right)\widetilde{H}_{k,2}\left(q\right)\;=\;&\left[n+1\right]_{q}\widetilde{H}_{n,2}\left(q\right)\widetilde{H}_{n}\left(q\right)\qquad(2.12)\\
&+\widetilde{H}_{n,2}\left(q\right)\left(\frac{1}{2}-\left[n+1\right]_{q}\right)-\frac{1}{2}\widetilde{H}_{n}^{2}\left(q\right)+\frac{1}{2}\left(q+1\right)\widetilde{H}_{n}\left(q\right).
\end{aligned}$$

Proof. Observed that

$$\sum_{k=1}^{n}\left[k\right]_{q}\widetilde{H}_{k}^{2}\left(q\right)+\left[n+1\right]_{q}\widetilde{H}_{n+1}^{2}\left(q\right)=\sum_{k=1}^{n+1}\left[k\right]_{q}\widetilde{H}_{k}^{2}\left(q\right)=\sum_{k=0}^{n}\left[k+1\right]_{q}\widetilde{H}_{k+1}^{2}\left(q\right).$$

From $\left[k+1\right]_{q}=\left[k\right]_{q}+q^{k}$, we have

$$\sum_{k=0}^{n}q^{k}\widetilde{H}_{k}^{2}\left(q\right)=\left[n+1\right]_{q}\widetilde{H}_{n+1}^{2}\left(q\right)-2q\sum_{k=0}^{n}q^{k}\widetilde{H}_{k}\left(q\right)-\sum_{k=1}^{n+1}\frac{q^{2k}}{\left[k\right]_{q}}.$$

By Lemma 2.3, we have the proof. Also, for proof of (2.12), using (1.2) and (2.5), the result is obtained. □

Lemma 2.7. *For $n \geq 1$, we have*

$$\sum_{k=1}^{n} \frac{q^{3k}}{[k]_q^2} = \widetilde{H}_{n,2}(q) - 2(1-q)\widetilde{H}_n(q) + q(1-q)^2 [n]_q, \tag{2.13}$$

and

$$\begin{aligned} &\sum_{k=1}^{n} \frac{q^{2k}}{[k]_q} \widetilde{H}_k(q) \\ = \ & \frac{1}{2}\widetilde{H}_{n,2}(q) + \frac{1}{2}\widetilde{H}_n(q)\left(\widetilde{H}_n(q) + 2q^{n+1} + q - 3\right) + [n]_q\, q(1-q). \end{aligned} \tag{2.14}$$

Proof. By (1.2) and (1.5), using Lemmas 2.5 and 2.6, the proof is complete. □

Lemma 2.8. *For $n \geq 1$, we have*

$$\begin{aligned} &\sum_{k=0}^{n} q^k \widetilde{H}_k^3(q) \\ = \ & [n+1]_q \widetilde{H}_{n+1}^3(q) - \frac{3}{2}(1+q)[n+1]_q \widetilde{H}_{n+1}^2(q) - \frac{1}{2}(1-q)^2 [n+1]_q \\ & + [n+1]_q \left(\widetilde{H}_{n+1}(q) - q\right)\left(\frac{7}{2}q - \frac{1}{2q} + q^2 + 2\right) - \frac{3}{2}q^{n+1}\widetilde{H}_{n+1}^2(q) \\ & + \frac{1}{2}\widetilde{H}_{n+1}(q)\left(q^n(1+q)(4q-1) + \frac{1}{q}(1-q)^2\right) \\ & + \frac{1}{2}\widetilde{H}_{n+1,2}(q)\left(q^{2n+2} + q^n\left(1-q^2\right)[n+1]_q + (1-q)^2 [n]_q [n+1]_q\right). \end{aligned} \tag{2.15}$$

Proof. Consider that

$$\begin{aligned} &\sum_{k=1}^{n} [k]_q \widetilde{H}_k^3(q) + [n+1]_q \widetilde{H}_{n+1}^3(q) \\ = \ & \sum_{k=1}^{n+1} [k]_q \widetilde{H}_k^3(q) = \sum_{k=0}^{n} [k+1]_q \widetilde{H}_{k+1}^3(q) \\ = \ & \sum_{k=0}^{n} [k+1]_q \left(\widetilde{H}_k(q) + \frac{q^{k+1}}{[k+1]_q}\right)^3 \\ = \ & \sum_{k=0}^{n} [k+1]_q \left(\widetilde{H}_k^3(q) + 3\widetilde{H}_k^2(q)\frac{q^{k+1}}{[k+1]_q} + 3\widetilde{H}_k(q)\frac{q^{2(k+1)}}{[k+1]_q^2} + \frac{q^{3(k+1)}}{[k+1]_q^3}\right) \\ = \ & \sum_{k=0}^{n} \left([k]_q + q^k\right)\widetilde{H}_k^3(q) + 3q\sum_{k=0}^{n} q^k \widetilde{H}_k^2(q) + 3\sum_{k=0}^{n} \widetilde{H}_k(q)\frac{q^{2(k+1)}}{[k+1]_q} + \sum_{k=0}^{n} \frac{q^{3(k+1)}}{[k+1]_q^2} \\ = \ & \sum_{k=0}^{n} [k]_q \widetilde{H}_k^3(q) + \sum_{k=0}^{n} q^k \widetilde{H}_k^3(q) + 3q\sum_{k=0}^{n} q^k \widetilde{H}_k^2(q) + 3\sum_{k=1}^{n+1} \widetilde{H}_{k-1}(q)\frac{q^{2k}}{[k]_q} + \sum_{k=1}^{n+1} \frac{q^{3k}}{[k]_q^2} \\ = \ & \sum_{k=0}^{n} [k]_q \widetilde{H}_k^3(q) + \sum_{k=0}^{n} q^k \widetilde{H}_k^3(q) + 3q\sum_{k=0}^{n} q^k \widetilde{H}_k^2(q) + 3\sum_{k=1}^{n+1} \widetilde{H}_k(q)\frac{q^{2k}}{[k]_q} - 2\sum_{k=1}^{n+1} \frac{q^{3k}}{[k]_q^2}. \end{aligned}$$

From here, (2.11) and (2.14) yield that

$$\sum_{k=0}^{n} q^k \widetilde{H}_k^3(q)$$

$$
\begin{aligned}
&= [n+1]_q \widetilde{H}_{n+1}^3(q) - 3q\sum_{k=1}^{n} q^k \widetilde{H}_k^2(q) - 3\sum_{k=1}^{n+1}\frac{q^{2k}}{[k]_q}\widetilde{H}_k(q) + 2\sum_{k=1}^{n+1}\frac{q^{3k}}{[k]_q^2} \\
&= -3q\left([n+1]_q\left(\widetilde{H}_{n+1}^2(q) - (1+q)\left(\widetilde{H}_{n+1}(q) - q\right)\right) - q^{n+1}\widetilde{H}_{n+1}(q)\right) \\
&\quad -\frac{3}{2}\widetilde{H}_{n+1,2}(q)\left(q^{2n+2} + [n+1]_q(1-q)\left(1+q^{n+1}\right)\right) - \frac{3}{2}\widetilde{H}_{n+1}^2(q) + 2\sum_{k=1}^{n+1}\frac{q^{3k}}{[k]_q^2} \\
&\quad -3[n+1]_q q(1-q) + [n+1]_q \widetilde{H}_{n+1}^3(q) - \frac{3}{2}\widetilde{H}_{n+1}(q)\left(2q^{n+2} + q - 3\right).
\end{aligned}
$$

By (2.13), we have the desired result. □

Lemma 2.9. *For $n \geq 1$, we have*

$$
\begin{aligned}
&\sum_{k=1}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) \qquad (2.16)\\
&= \frac{1}{(1+q)^2}\left([3]_q q(q^n-1) - \frac{q^{2n+2}}{q-1}\right) \\
&\quad - \frac{1}{1+q}\widetilde{H}_{n,2}(q)\left(\frac{1}{1-q^2} + q[n]_q\right) - \frac{1}{1+q}\left(\widetilde{H}_n^2(q)\right. \\
&\quad \left. - [2n+2]_q \widetilde{H}_n(q)\widetilde{H}_{n,2}(q) - \widetilde{H}_n(q)\left(\frac{2}{1+q}[3]_q - q^{n+1}\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{k=0}^{n} q^k [k]_q \widetilde{H}_k(q)\widetilde{H}_{k,2}(q) \qquad (2.17)\\
&= \frac{q}{q+1}[n]_q[n+1]_q\widetilde{H}_n(q)\widetilde{H}_{n,2}(q) + q\left(1 - \frac{q}{(q+1)^2}\right)[n]_q \\
&\quad + \frac{1}{2(1+q)}\widetilde{H}_n^2(q) - \left(\frac{1}{(q+1)^2} + \frac{q}{q+1}[n]_q + \frac{1}{2}\right)\widetilde{H}_n(q) \\
&\quad + \frac{q^{2n+2}}{2(1+q)^2(q-1)} - \frac{1}{2(q+1)}\left(q^2[n]_q^2 - \frac{1}{(1-q^2)}\right)\widetilde{H}_{n,2}(q).
\end{aligned}
$$

Proof. Firstly, by (1.5) and (2.12), we have

$$
\begin{aligned}
&\sum_{k=1}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) \\
&= (1-q)\sum_{k=1}^{n-1} q^k [k+1]_q \widetilde{H}_{k+1}(q)\widetilde{H}_{k,2}(q) \\
&\quad -q(1-q)\sum_{k=1}^{n-1} q^k [k+1]_q \widetilde{H}_{k,2}(q) - \frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^{2k}\widetilde{H}_{k,2}(q) \\
&\quad - \frac{1}{2}(1-q)^2\sum_{k=1}^{n-1} q^k [k]_q \widetilde{H}_{k,2}(q) + \frac{1}{2}(1-q)^2\sum_{k=1}^{n-1} q^k \widetilde{H}_k(q) \\
&\quad - \frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^k \widetilde{H}_k^2(q) + q(1-q)\sum_{k=1}^{n-1} q^k \widetilde{H}_k(q) \\
&\quad - \frac{1}{2}q^n\left(q^n\widetilde{H}_{n,2}(q) + (1-q)\left([n]_q\widetilde{H}_{n,2}(q) - \widetilde{H}_n(q)\right) + \widetilde{H}_n^2(q)\right) \\
&\quad + q^n[n+1]_q\left(\widetilde{H}_{n+1}(q) - q\right)\widetilde{H}_{n,2}(q) + q^{n+1}\widetilde{H}_n(q)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1-q}{q}\sum_{k=1}^{n} q^k [k]_q \widetilde{H}_k(q)\widetilde{H}_{k,2}(q) - \frac{1-q}{q}\sum_{k=1}^{n}\frac{q^{2k}}{[k]_q}\widetilde{H}_k(q) \\
& -\frac{1}{2}(1-q)(3-q)\sum_{k=1}^{n-1} q^k [k]_q \widetilde{H}_{k,2}(q) - \frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^{2k}\widetilde{H}_{k,2}(q) \\
& \quad + (1-q)\sum_{k=1}^{n}\frac{q^{2k}}{[k]_q} + \frac{1}{2}\left(1-q^2\right)\sum_{k=1}^{n-1} q^k \widetilde{H}_k(q) \\
& \quad - \frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^k \widetilde{H}_k^2(q) - (1-q)q^n [n]_q \widetilde{H}_{n,2}(q) \\
& \quad - \frac{1}{2}q^n\left(q^n\widetilde{H}_{n,2}(q) + (1-q)\left([n]_q\widetilde{H}_{n,2}(q) - \widetilde{H}_n(q)\right) + \widetilde{H}_n^2(q)\right) \\
& \quad + q^n [n+1]_q\left(\widetilde{H}_{n+1} - q\right)\widetilde{H}_{n,2}(q) + q^{n+1}\widetilde{H}_n(q). \qquad (2.18)
\end{aligned}
$$

Secondly, we can write

$$
\begin{aligned}
& \sum_{k=1}^{n}[k]_q^2\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) + [n+1]_q^2\widetilde{H}_{n+1}(q)\widetilde{H}_{n+1,2}(q) \\
= \quad & \sum_{k=1}^{n+1}[k]_q^2\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) = \sum_{k=0}^{n}[k+1]_q^2\widetilde{H}_{k+1}(q)\widetilde{H}_{k+1,2}(q) \\
= \quad & \sum_{k=0}^{n}[k+1]_q^2\left(\widetilde{H}_k(q) + \frac{q^{k+1}}{[k+1]_q}\right)\left(\widetilde{H}_{k,2}(q) + \frac{q^{k+1}}{[k+1]_q^2}\right) \\
= \quad & \sum_{k=0}^{n}[k+1]_q^2\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) + \sum_{k=0}^{n} q^{k+1}\widetilde{H}_k(q) \\
& \qquad + \sum_{k=0}^{n} q^{k+1}[k+1]_q\widetilde{H}_{k,2}(q) + \sum_{k=0}^{n}\frac{q^{2k+2}}{[k+1]_q} \\
= \quad & \sum_{k=0}^{n}[k]_q^2\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) + 2\sum_{k=0}^{n} q^k [k]_q\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) + \sum_{k=0}^{n}\frac{q^{2k+2}}{[k+1]_q} \\
& + \sum_{k=0}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) + \sum_{k=0}^{n} q^{k+1}\widetilde{H}_k(q) + \sum_{k=0}^{n} q^{k+1}[k+1]_q\widetilde{H}_{k,2}(q).
\end{aligned}
$$

After from some combinatorial operations, we have

$$
\begin{aligned}
& \sum_{k=0}^{n} q^k [k]_q\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) \qquad (2.19) \\
= \quad & \frac{1}{2}[n+1]_q^2\widetilde{H}_{n+1}(q)\widetilde{H}_{n+1,2}(q) \\
& -\frac{1}{2}\sum_{k=0}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) - \frac{1}{2}q\sum_{k=0}^{n} q^k\widetilde{H}_k(q) - \frac{1}{2}\sum_{k=1}^{n+1} q^k [k]_q\widetilde{H}_{k,2}(q).
\end{aligned}
$$

Substituting (2.19) in (2.18), we have

$$
\begin{aligned}
& \sum_{k=1}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) \\
= \quad & -\frac{1-q}{2q}\sum_{k=0}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) - \frac{1-q}{2q}\sum_{k=1}^{n+1} q^k [k]_q\widetilde{H}_{k,2}(q) - \frac{1-q}{2}\sum_{k=0}^{n} q^k\widetilde{H}_k(q)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2}(1-q)(3-q)\sum_{k=1}^{n-1} q^k [k]_q \widetilde{H}_{k,2}(q) - \frac{1-q}{q}\sum_{k=1}^{n} \frac{q^{2k}}{[k]_q}\widetilde{H}_k(q) + (1-q)\sum_{k=1}^{n}\frac{q^{2k}}{[k]_q} \\
& -\frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^{2k}\widetilde{H}_{k,2}(q) + \frac{1}{2}\left(1-q^2\right)\sum_{k=1}^{n-1} q^k \widetilde{H}_k(q) - \frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^k \widetilde{H}_k^2(q) \\
& +q^n [n+1]_q \left(\widetilde{H}_{n+1}(q)-q\right)\widetilde{H}_{n,2}(q) + q^{n+1}\widetilde{H}_n(q) - (1-q) q^n [n]_q \widetilde{H}_{n,2}(q) \\
& -\frac{1}{2}q^n\left(q^n\widetilde{H}_{n,2}(q) + (1-q)\left([n]_q\widetilde{H}_{n,2}(q) - \widetilde{H}_n(q)\right) + \widetilde{H}_n^2(q)\right) \\
& +\frac{1-q}{2q}[n+1]_q^2 \widetilde{H}_{n+1}(q)\widetilde{H}_{n+1,2}(q) \\
= & \frac{1-q}{2q}[n+1]_q^2 \widetilde{H}_{n+1}(q)\widetilde{H}_{n+1,2}(q) - \frac{1-q}{2q}\sum_{k=0}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) \\
& -\frac{1-q}{q}\sum_{k=1}^{n}\frac{q^{2k}}{[k]_q}\widetilde{H}_k(q) - \frac{1}{2q}(1-q)\left(3q-q^2+1\right)\sum_{k=1}^{n-1} q^k [k]_q \widetilde{H}_{k,2}(q) \\
& +(1-q)\sum_{k=1}^{n}\frac{q^{2k}}{[k]_q} - \frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^{2k}\widetilde{H}_{k,2}(q) + \frac{1}{2}q(1-q)\sum_{k=1}^{n-1} q^k\widetilde{H}_k(q) \\
& -\frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^k\widetilde{H}_k^2(q) + q^n[n+1]_q\left(\widetilde{H}_{n+1}-q\right)\widetilde{H}_{n,2}(q) \\
& -\frac{1-q}{2}q^n\widetilde{H}_n(q) - \frac{1-q}{2q}\left(q^n[n]_q\widetilde{H}_{n,2}(q) + q^{n+1}[n+1]_q\widetilde{H}_{n+1,2}(q)\right) \\
& -\frac{1}{2}q^n\left(q^n\widetilde{H}_{n,2}(q) + (1-q)\left([n]_q\widetilde{H}_{n,2}(q) - \widetilde{H}_n(q)\right) + \widetilde{H}_n^2(q)\right) \\
& -(1-q)q^n[n]_q\widetilde{H}_{n,2}(q) + q^{n+1}\widetilde{H}_n(q).
\end{aligned}
$$

From (1.2), (2.3), (2.14) and Lemma 2.6, we get

$$
\begin{aligned}
& \left(1+\frac{1-q}{2q}\right)\sum_{k=0}^{n} q^{2k}\widetilde{H}_k(q)\widetilde{H}_{k,2}(q) \\
= & \frac{1}{2q}(1-q)\left(q^2-3q-1\right)\sum_{k=1}^{n-1} q^k[k]_q\widetilde{H}_{k,2}(q) - \frac{1}{2}(1-q)\sum_{k=1}^{n-1} q^{2k}\widetilde{H}_{k,2}(q) \\
& -\frac{1-q}{q}\left(\frac{1}{2}\widetilde{H}_{n,2}(q) + \frac{1}{2}\widetilde{H}_n^2(q) + [n]_q q(1-q) + \frac{1}{2}\widetilde{H}_n(q)\left(2q^{n+1}+q-3\right)\right) \\
& +(1-q)q^n\widetilde{H}_n(q) + (1-q)^2[n]_q\left(\widetilde{H}_n(q)-q\right) - (1-q)q^n[n]_q\widetilde{H}_{n,2}(q) \\
& -\frac{1}{2}(1-q)\left([n]_q\widetilde{H}_n^2(q) - q^n\widetilde{H}_n(q) - (1+q)[n]_q\left(\widetilde{H}_n(q)-q\right)\right) \\
& +\frac{1}{2}q(1-q)[n]_q\left(\widetilde{H}_n(q)-q\right) + q^n[n+1]_q\left(\widetilde{H}_{n+1}(q)-q\right)\widetilde{H}_{n,2}(q) \\
& -\frac{1-q}{2q}\left(q^n[n]_q\widetilde{H}_{n,2}(q) + q^{n+1}[n+1]_q\widetilde{H}_{n+1,2}(q)\right) \\
& +\frac{1-q}{2q}[n+1]_q^2\widetilde{H}_{n+1}(q)\widetilde{H}_{n+1,2}(q) + q^{n+1}\widetilde{H}_n(q) - \frac{1-q}{2}q^n\widetilde{H}_n(q) \\
& -\frac{1}{2}q^n\left(q^n\widetilde{H}_{n,2}(q) + (1-q)\left([n]_q\widetilde{H}_{n,2}(q) - \widetilde{H}_n(q)\right) + \widetilde{H}_n^2(q)\right),
\end{aligned}
$$

and with the help of (2.7) and (2.8), we have the proofs of the two sums. □

Lemma 2.10. *For $n \geq 1$, we have*

$$\begin{aligned}
&\sum_{k=1}^{n} q^{2k} [k]_q \widetilde{H}_{k,2}(q) \\
= \; & \frac{1}{[3]_q} \left(q^2 [n]_q [n+1]_q \left(q^n + \frac{1}{q+1} \right) \widetilde{H}_{n+1,2}(q) \right. \\
& \left. + \left(2 - \frac{1}{1+q} \right) \widetilde{H}_{n+1}(q) + \frac{q^2}{1+q} [n+1]_q \left(q(1-q)[n]_q - 1 \right) \right),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{n} q^{k} [k]_q^2 \widetilde{H}_{k,2}(q) \\
= \; & \frac{1}{(1+q)[3]_q} \left(q[n]_q [n+1]_q \left([n+1]_q + q^2 [n]_q \right) \widetilde{H}_{n+1,2}(q) \right. \\
& \left. - q\widetilde{H}_{n+1}(q) + q^2 [n+1]_q \left(1 - q[n]_q \right) \right).
\end{aligned}$$

Proof. By (1.5) and (2.8), the proof is similar to the proof of Lemma 2.9. □

Theorem 2.11. *For any prime number p and $\alpha \in \mathbb{Z}^+$, we have*

$$\begin{aligned}
&\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk + \binom{k+1}{2} + k} \begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \\
\equiv \; & [p]_q - 1 - [p]_q [\alpha]_{q^p} q^{-\alpha p} \left(q^p + [p]_q \left(\frac{p-1}{2}(1-q) - q \right) \right) \\
& + \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2p\alpha} (1-q) \\
& \times \left(q^{p-1} \left(\frac{q(3-p)}{2} + \frac{p-1}{2} + q \right) - \frac{(p-1)(1-q)}{2q(1+q)} \right) \pmod{[p]_q^3}.
\end{aligned}$$

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk + \binom{k+1}{2} + k} \begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \\
= \; & \sum_{k=1}^{p-1} q^k \left(1 - [p]_q [\alpha]_{q^p} q^{-\alpha p} \widetilde{H}_k(q) \right. \\
& \left. + \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 \left(\widetilde{H}_k^2(q) - q^k \widetilde{H}_{k,2}(q) - (1-q) \left([k]_q \widetilde{H}_{k,2}(q) - \widetilde{H}_k(q) \right) \right) \right),
\end{aligned}$$

and with the help of Lemmas 2.5 and 2.6, then

$$\begin{aligned}
&\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk + \binom{k+1}{2} + k} \begin{bmatrix} \alpha p - 1 \\ k \end{bmatrix}_q \\
\equiv \; & [p]_q - 1 - [p]_q [\alpha]_{q^p} q^{-\alpha p} \left(q^p - [p]_q \left(\frac{(p-1)(1-q)}{2} + q \right) \right) \\
& + \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2p\alpha} \left(q^p \frac{(1-q)(3-p)}{2} + \frac{1-q}{1+q} q^p - \frac{(p-1)(1-q)^2}{2q(1+q)} \right. \\
& \left. + \frac{(p-1)(1-q)}{2} q^{p-1} + \frac{1-q}{1+q} q^{p+1} \right) \pmod{[p]_q^3}.
\end{aligned}$$

Using some combinatorial operations, we have the proof. □

Theorem 2.12. *For any prime number p and $\alpha \in \mathbb{Z}^+$, we have*

$$\begin{aligned} &\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk+\binom{k+1}{2}+k} [k]_q \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q \\ \equiv\ & \frac{q^{1-\alpha p}}{(1-q^2)^2} \left(q^{\alpha p+2} (q^p-2) + q^{\alpha p} - q^p + q^2 \right) [p-1]_q \pmod{[p]_q^3}. \end{aligned}$$

Proof. By (1.4), Lemmas 2.4 and 2.10, the proof is similar to the proof of Theorem 2.11. □

Theorem 2.13. *For any prime number p and $\alpha \in \mathbb{Z}^+$, we have*

$$\begin{aligned} &\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk+\binom{k+1}{2}+k} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q \widetilde{H}_k(q) \\ \equiv\ & \left(q(2+q) - \left(3q^2+4q-1\right) q^{p\alpha} + 3q^{2p\alpha+2} \right) \frac{q-q^p}{q^{2p\alpha}(q-1)^3} \\ &+ \frac{\widetilde{H}_{p-1}(q)}{2q^{2p\alpha}(q-1)^3} \left(q^{2p\alpha} \left(5q^{p+1} + 2q^{p+2} - q^p - 6q^2 - q + 1 \right) \right. \\ &\left. + 2q^{p\alpha} \left(5q - 7q^{p+1} + 3q^2 + q^p - 2 \right) - 5q + 5q^{p+1} - 2q^2 + q^p + 1 \right) \\ &- \frac{3q - q^{p+1} - 2q^p + q^{p\alpha}\left(3q^{p+1} - 5q + 2\right)}{2q^{2p\alpha}(q-1)^3} \left(q^{p\alpha} - 1\right) \widetilde{H}_{p-1}^2(q) \pmod{[p]_q^3}. \end{aligned}$$

Proof. From (2.1), we have

$$\begin{aligned} &\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk+\binom{k+1}{2}+k} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q \widetilde{H}_k(q) \\ \equiv\ & \sum_{k=1}^{p-1} q^k \widetilde{H}_k(q) \left(1 - [p]_q [\alpha]_{q^p} q^{-\alpha p} \widetilde{H}_k(q) + \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \right. \\ &\left. \times \left(\widetilde{H}_k^2(q) - q^k \widetilde{H}_{k,2}(q) - (1-q) \left([k]_q \widetilde{H}_{k,2}(q) - \widetilde{H}_k(q) \right) \right) \right) \\ =\ & \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \sum_{k=1}^{p-1} q^k \widetilde{H}_k^3(q) - [p]_q [\alpha]_{q^p} q^{-\alpha p} \sum_{k=1}^{p-1} q^k \widetilde{H}_k^2(q) \\ &+ \sum_{k=1}^{p-1} q^k \widetilde{H}_k(q) + \frac{1}{2}(1-q) [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \sum_{k=1}^{p-1} q^k \widetilde{H}_k^2(q) \\ &- \frac{1-q}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \sum_{k=1}^{p-1} q^k [k]_q \widetilde{H}_k(q) \widetilde{H}_{k,2}(q) \\ &- \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \sum_{k=1}^{p-1} q^{2k} \widetilde{H}_k(q) \widetilde{H}_{k,2}(q) \pmod{[p]_q^3}. \end{aligned}$$

With the help of (1.2), (2.11), (2.15) and Lemma 2.9, we have

$$\begin{aligned} &\sum_{k=1}^{p-1} (-1)^k q^{-\alpha pk+\binom{k+1}{2}+k} \begin{bmatrix} \alpha p-1 \\ k \end{bmatrix}_q \widetilde{H}_k(q) \\ \equiv\ & [p]_q \left(\widetilde{H}_p(q) - q \right) - [p]_q^2 [\alpha]_{q^p} q^{-\alpha p} \left(\widetilde{H}_p^2(q) - (1+q) \left(\widetilde{H}_p(q) - q \right) \right) \\ &+ [p]_q [\alpha]_{q^p} q^{-\alpha p+p} \widetilde{H}_p(q) + \frac{1}{2} [p]_q^2 [\alpha]_{q^p}^2 q^{-2\alpha p} \left([p]_q \widetilde{H}_p^3(q) - \frac{3}{2}(1+q) [p]_q \widetilde{H}_p^2(q) \right. \end{aligned}$$

$$
\begin{aligned}
&+[p]_q\left(\widetilde{H}_p(q)-q\right)\left(\frac{7}{2}q-\frac{1}{2q}+q^2+2\right)-\frac{1}{2}(1-q)^2[p]_q-\frac{3}{2}q^p\widetilde{H}_p^2(q)\\
&+\frac{1}{2}\widetilde{H}_p(q)\left(q^{p-1}(1+q)(4q-1)+\frac{1}{q}(1-q)^2\right)\\
&+\frac{1}{2}\widetilde{H}_{p,2}(q)\left(q^{2p}+q^{p-1}\left(1-q^2\right)[p]_q+(1-q)^2[p-1]_q[p]_q\right)\Big)\\
&-\frac{1}{4}[p]_q^2[\alpha]_{q^p}^2q^{-2\alpha p}\left(\left(2\widetilde{H}_{p-1}(q)-q\right)\left(q+\frac{1}{1+q}\right)-\widetilde{H}_{p-1}^2(q)\right.\\
&-\frac{1}{q-1}\widetilde{H}_{p-1}(q)\widetilde{H}_{p-1,2}(q)+\frac{q^{2p}}{q-1}\widetilde{H}_{p-1,2}(q)\left(\widetilde{H}_{p-1}(q)-\frac{1}{1+q}\right)\\
&+\frac{1+q}{2}\widetilde{H}_{p-1,2}(q)\left(\frac{3}{2(q-1)}+\frac{1}{2(1+q)}-\frac{1}{(1+q)^2}\right)\\
&\left.+q^p\left(q+\frac{1}{1+q}-\widetilde{H}_{p-1}(q)-\frac{1}{q-1}\widetilde{H}_{p-1,2}(q)\right)\right)\\
&+\frac{1}{2}(1-q)[p]_q^2[\alpha]_{q^p}^2q^{-2\alpha p}\left([p]_q\widetilde{H}_p^2(q)-q^p\widetilde{H}_p(q)\right)\\
&-\frac{1}{2}\left(1-q^2\right)[p]_q^3[\alpha]_{q^p}^2q^{-2\alpha p}\left(\widetilde{H}_p(q)-q\right)\\
&-\frac{1-q}{4}[p]_q^2[\alpha]_{q^p}^2q^{-2\alpha p}\left([p]_q^2\widetilde{H}_p(q)\widetilde{H}_{p,2}(q)-q[p]_q\left(\widetilde{H}_p(q)-q\right)\right.\\
&\left.-\frac{1}{1+q}\left(q[p]_q[p+1]_q\widetilde{H}_{p+1,2}(q)+\widetilde{H}_{p+1}(q)-q[p+1]_q\right)\right)\pmod{[p]_q^3},
\end{aligned}
$$

and with the help of the congruences $[p]_q\widetilde{H}_p^3(q)\equiv 3q^p\widetilde{H}_{p-1}^2(q)+3\frac{q^{2p}}{[p]_q}\widetilde{H}_{p-1}(q)+\frac{q^{3p}}{[p]_q^2}\pmod{[p]_q}$ and $[p]_q\widetilde{H}_p^2(q)\equiv 2q^p\widetilde{H}_{p-1}(q)+\frac{q^{2p}}{[p]_q}\pmod{[p]_q}$,

$$
\begin{aligned}
&\sum_{k=1}^{p-1}(-1)^k q^{-\alpha pk+\binom{k+1}{2}+k}\begin{bmatrix}\alpha p-1\\ k\end{bmatrix}_q\widetilde{H}_k(q)\\
\equiv\;&[p]_q\left(\widetilde{H}_p(q)-q\right)+[p]_q[\alpha]_{q^p}q^{-\alpha p+p}\widetilde{H}_p(q)-[p]_q^2[\alpha]_{q^p}q^{-\alpha p}\left(\widetilde{H}_p^2(q)\right.\\
&\left.-(1+q)\left(\widetilde{H}_p(q)-q\right)\right)+\frac{1}{2}[p]_q^2[\alpha]_{q^p}^2q^{-2\alpha p}\left(3q^p\widetilde{H}_{p-1}^2(q)+\frac{q^{3p}}{[p]_q^2}\right.\\
&+3\frac{q^{2p}}{[p]_q}\widetilde{H}_{p-1}(q)-\frac{3}{2}(1+q)[p]_q\widetilde{H}_p^2(q)+[p]_q\widetilde{H}_p(q)\left(\frac{7}{2}q-\frac{1}{2q}+q^2+2\right)\\
&-\frac{3}{2}\left(2q^p\widetilde{H}_{p-1}(q)+\frac{q^{2p}}{[p]_q}\right)+\frac{1}{2}\widetilde{H}_p(q)\left(q^{p-1}(1+q)(4q-1)+\frac{1}{q}(1-q)^2\right)\\
&\left.+\frac{1}{2}q^{2p}\widetilde{H}_{p,2}(q)+\frac{1}{2}[p]_q\widetilde{H}_{p,2}(q)\left(q^{p-1}\left(1-q^2\right)+(1-q)^2[p-1]_q\right)\right)\\
&-\frac{1}{4}[p]_q^2[\alpha]_{q^p}^2q^{-2\alpha p}\left(\left(q+\frac{1}{1+q}\right)\left(2\widetilde{H}_{p-1}(q)-q\right)-\widetilde{H}_{p-1}^2(q)\right.\\
&-\frac{1}{q-1}\widetilde{H}_{p-1}(q)\widetilde{H}_{p-1,2}(q)+\frac{q^{2p}}{q-1}\widetilde{H}_{p-1,2}(q)\left(\widetilde{H}_{p-1}(q)-\frac{1}{1+q}\right)\\
&+\frac{1+q}{2}\widetilde{H}_{p-1,2}(q)\left(\frac{3}{2(q-1)}+\frac{1}{2(1+q)}-\frac{1}{(1+q)^2}\right)\\
&\left.+q^p\left(q+\frac{1}{1+q}-\widetilde{H}_{p-1}(q)-\frac{1}{q-1}\widetilde{H}_{p-1,2}(q)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}(1-q)[p]_q^2[\alpha]_{q^p}^2 q^{-2\alpha p}\widetilde{H}_p(q)\left([p]_q\widetilde{H}_p(q)-q^p-(1+q)[p]_q\right)\\
&-\frac{1}{4}(1-q)[p]_q^2[\alpha]_{q^p}^2 q^{-2\alpha p}\left([p]_q^2\widetilde{H}_p(q)\widetilde{H}_{p,2}(q)-q[p]_q\widetilde{H}_p(q)\right.\\
&\left.-\frac{1}{(1+q)}\left(q[p+1]_q\left([p]_q\widetilde{H}_{p+1,2}(q)-1\right)+\widetilde{H}_{p+1}(q)\right)\right) \pmod{[p]_q^3}.
\end{aligned}
$$

Again, using congruences $[p]_q^2\widetilde{H}_p(q)\widetilde{H}_{p,2}(q) \equiv q^p\widetilde{H}_{p-1}(q)+\frac{q^{2p}}{[p]_q} \pmod{[p]_q}$ and $[p]_q[p+1]_q\widetilde{H}_{p+1,2}(q)\equiv\frac{[p+1]_q}{[p]_q}q^p+\frac{[p]_q}{[p+1]_q}q^{p+1} \pmod{[p]_q}$, we can write

$$
\begin{aligned}
&\sum_{k=1}^{p-1}(-1)^k q^{-\alpha pk+\binom{k+1}{2}+k}\begin{bmatrix}\alpha p-1\\ k\end{bmatrix}_q\widetilde{H}_k(q)\\
\equiv\ &[p]_q\left(\widetilde{H}_p(q)-q\right)+[p]_q[\alpha]_{q^p}q^{-\alpha p+p}\widetilde{H}_p(q)-[p]_q^2[\alpha]_{q^p}q^{-\alpha p}\left(\widetilde{H}_p^2(q)\right.\\
&\left.-(1+q)\left(\widetilde{H}_p(q)-q\right)\right)+\frac{1}{2}[p]_q^2[\alpha]_{q^p}^2q^{-2\alpha p}\left(3q^p\widetilde{H}_{p-1}^2(q)+\frac{q^{3p}}{[p]_q^2}\right.\\
&+3\frac{q^{2p}}{[p]_q}\widetilde{H}_{p-1}(q)-\frac{3}{2}(1+q)\left(2q^p\widetilde{H}_{p-1}(q)+\frac{q^{2p}}{[p]_q}\right)-\frac{3}{2}q^p\widetilde{H}_p^2(q)\\
&+q^p\left(\frac{7}{2}q-\frac{1}{2q}+q^2+2\right)+\frac{1}{2}\widetilde{H}_p(q)\left(q^{p-1}(1+q)(4q-1)+\frac{1}{q}(1-q)^2\right)\\
&+\frac{1}{2}q^{2p}\widetilde{H}_{p,2}(q)+\frac{q^p}{2[p]_q}\left(q^{p-1}\left(1-q^2\right)+(1-q)^2[p-1]_q\right)\\
&-\frac{1}{2}\left(\left(2\widetilde{H}_{p-1}(q)-q\right)\left(q+\frac{1}{1+q}\right)-\frac{1}{q-1}\widetilde{H}_{p-1}(q)\widetilde{H}_{p-1,2}(q)\right.\\
&+\frac{1+q}{2}\widetilde{H}_{p-1,2}(q)\left(\frac{3}{2(q-1)}+\frac{1}{2(1+q)}-\frac{1}{(1+q)^2}\right)-\widetilde{H}_{p-1}^2(q)\\
&+q^p\left(q+\frac{1}{1+q}-\widetilde{H}_{p-1}(q)-\frac{1}{q-1}\widetilde{H}_{p-1,2}(q)\right)\\
&\left.+\frac{q^{2p}}{q-1}\widetilde{H}_{p-1,2}(q)\left(\widetilde{H}_{p-1}(q)-\frac{1}{1+q}\right)\right)\\
&+\frac{1-q}{2(1+q)}\left(q^{p+1}\frac{[p+1]_q}{[p]_q}+q^{p+2}\frac{[p]_q}{[p+1]_q}+\widetilde{H}_{p+1}(q)-q[p+1]_q\right)\\
&+\frac{1}{2}(1-q)\left(q^{p+1}-q^p\widetilde{H}_{p-1}(q)-\frac{q^{2p}}{[p]_q}\right)\\
&\left.+(1-q)\left(2q^p\widetilde{H}_{p-1}(q)+\frac{q^{2p}}{[p]_q}-q^p\widetilde{H}_p(q)-q^p(1+q)\right)\right) \pmod{[p]_q^3}.
\end{aligned}
$$

Thus, the proof is complete. □

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