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SOME RESULTS ON PSEUDOSYMMETRIC NORMAL PARACONTACT METRIC MANIFOLDS

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Abstract. In this article, the M-projective and Weyl curvature tensors on a normal paracontact metric manifold are discussed. For normal paracontact metric manifolds, pseudosymmetric cases are investigated and some interesting results are obtained. We show that a semisymmetric normal paracontact manifold is of constant sectional curvature. We also obtain that a pseudosymmetric normal paracontact metric manifold is an $\eta-$Einstein manifold. Finally, we support our topic with an example.

1. Introduction

The notion of odd-dimensional manifolds with contact and almost contact structures was initiated by Boothby and Wang [1]. In [2], Sasaki and Hatakeyama reinvestigated them using tensor calculus. Tanno in [3] classified connected almost contact metric manifolds whose automorphism groups possess maximum dimension. For such manifolds, the sectional curvature of a plane section containing ξ is a constant named c. He showed that it can be divided into the following three classes.

- **Class-1**$\Rightarrow$ Homogeneous normal contact Riemannian manifolds with $c > 0$.

- **Class-2**$\Rightarrow$ Global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$.

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- **Class-3**⇒ A warped product space $\mathbb{R} \times_f C$ if $c < 0$.

It is well known that the manifolds of class-1 are characterized by admitting a Sasakian structure. In [4], Kenmotsu investigated the differential geometric properties of the manifolds of class-3. In general, these structures are not Sasakian [5].

In [6], S. Zamkovoy and G. Nakova reviewed the decomposition of almost contact metric manifolds in eleven classes. In addition to almost paracontact metric manifolds, K. Mandal and U.C De in [7], N. Özdemir, S. Aktay and M.solgun in [8] examined paracontact metric manifolds and obtained their various geometric properties. Also, in [9], H. Pandey and A. Kumar examined the anti-invariant submanifolds of almost paracontact manifolds. Similarly, J. Welyczko [10] studied Legendre curves on 3-dimensional normal paracontact metric manifolds.

After then, in [11], Pokhariyal and Mishra have introduced an $\mathcal{M}$-projective curvature tensor on a Riemannian manifold. The properties of the $\mathcal{M}$-projective curvature tensor in Sasakian and Kähler manifolds were developed by Ojha in [12]. He showed that it bridges the gap between conformal curvature tensor, conharmonic curvature tensor, and concircular curvature tensor. $\mathcal{M}$-projective curvature tensor on different manifolds studied by many geometers such as Kenmotsu, Sasakian, and generalized Sasakian space form.

In [14], by using some tensors, invariant submanifolds of an almost Kenmotsu (κ, μ, ν)-space are characterized. Similarly, many authors have presented important work with various manifolds and some curvature tensors on them ([13], [15]- [18]).

Motivated by these ideas, we have attempted to study properties of the Weyl-conformal curvature tensor in a normal paracontact metric manifold. We think that some interesting results contribute differential geometry.

The present paper is organized as follows.

In section 2, we give the notations and preliminary results of normal paracontact metric manifolds that will be used later. In section 3, we show that a normal paracontact metric manifold satisfying $R(X,Y)\cdot R = 0$ if and only if it has constant sectional curvature and $R(X,Y)\cdot \mathcal{M} = 0$ implies that it η-Einstein manifold.

2. Preliminaries

An almost paracontact structure on a n-dimensional smooth manifold M is given by a $(1,1)$-type tensor field φ, a vector field ξ, and a 1-form η satisfying the following condition

$$\varphi^2 = I - \eta \otimes \xi, \;\; \eta(\xi) = 1. \tag{1}$$

As an immediate consequent $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and the tensor φ has constant rank $n-1$. If an almost paracontact manifold is endowed with a semi-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2}$$

for any $X, Y \in \Gamma(TM)$, then $M^n(\varphi, \xi, \eta, g)$ is called an almost paracontact metric manifold, where $\Gamma(TM)$ is the set of the differentiable vector fields on M.
It follows that

$$g(\varphi X, Y) = -g(X, \varphi Y).$$

The fundamental 2-form of the almost paracontact metric manifold is given by

$$\Phi(X, Y) = g(\varphi X, Y).$$

If $d\eta = \Phi$, then η becomes a contact form, that is, $\eta \wedge (d\eta)^n \neq 0$ and $M^n(\varphi, \xi, \eta, g)$ is said to be a paracontact metric manifold. Any such pseudo-Riemannian metric manifold is of signature $(\frac{n+1}{2}, \frac{n-1}{2})$ for $n = 2m+1$. In this case, we have

$$(\nabla_X \varphi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{3}$$

for any $X, Y \in \Gamma(TM)$, where ∇ denote the Levi-Civita connection on M. (1) and (3) imply that

$$\nabla_X \xi = \varphi X.$$

An almost paracontact structure is said to be normal if the tensor $N_\varphi - 2d\eta \oplus \xi = 0$ [13], where N_φ the Nijenhuis tensor of φ given by

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

For the sake of brevity, a normal paracontact metric manifold is said to be paracontact metric manifold [8].

A normal paracontact metric manifold M is of a constant sectional curvature c, then its Riemannian curvature tensor R is given by

$$\begin{aligned} R(X, Y)Z &= \frac{c+1}{4}\Big\{g(Y, Z)X - g(X, Z)Y\Big\} \\ &+ \frac{c-1}{4}\Big\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi \\ &- g(Y, Z)\eta(X)\xi + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Y\Big\}, \end{aligned} \tag{4}$$

for any $X, Y, Z \in \Gamma(TM)$ [8].

For a $(0, k)$-type tensor field T and a $(0, 2)$-type tensor field A on a semi-Riemannian manifold (M, g), the Tachibana tensor $Q(A, T)$ is defined as

$$\begin{aligned} Q(A, T)(X_1, X_2, ..., X_k; X, Y) &= -T((X \wedge_A Y)X_1, X_2, ...X_k) \\ &\quad -T(X_1, (X \wedge_A Y)X_2, ...X_k) \\ &\quad . \\ &\quad . \\ &\quad . \\ &- T(X_1, ...X_{k-1}, (X \wedge_A Y)X_k), \end{aligned} \tag{5}$$

for all $X_1, X_2, ...X_k, X, Y \in \Gamma(TM)$, where $X \wedge_A Y$ is an endomorphism defined by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y. \tag{6}$$

A semi-Riemannian manifold (M, g) is pseudosymmetric if its the Riemannian curvature tensor R satisfies

$$R \cdot R = LQ(g, R), \tag{7}$$

where L is a function on M. Particularly, if $L = 0$, it is called a semisymmetric manifold.

On a normal paracontact metric manifold M^n, the following relations hold;

$$\begin{aligned} R(X,Y)\xi &= \eta(X)Y - \eta(Y)X & (8)\\ R(\xi,X)Y &= \eta(Y)X - g(X,Y)\xi & (9)\\ \eta(R(X,Y)Z) &= \eta(Y)g(X,Z) - \eta(X)g(Y,Z) & (10)\\ S(X,\xi) &= (1-n)\eta(X), \quad Q\xi = (1-n)\xi, & (11) \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$, where S and Q are, respectively, the Ricci tensor and Ricci operatory of M given by $g(QX, Y) = S(X, Y)$.

On the other hand, the Weyl-conformal curvature and M-projective curvature tensors play an important role in differential geometry as well as in relativity. The Weyl-conformal curvature tensor and M-projective curvature tensor of a Riemannian manifold (M^n, g), $n > 2$, are respectively, defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY \\ &+ S(Y,Z)X - S(X,Z)Y\} \\ &+ \frac{\tau}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \mathcal{M}(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}\{S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY\}, \end{aligned} \tag{13}$$

for any $X, Y, Z \in \Gamma(TM)$, where τ denote the scalar curvature of M.

A normal paracontact metric manifold M is called η-Einstein if its Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{14}$$

for any $X, Y \in \Gamma(TM)$, where a and b are arbitrary constants. If $b = 0$, then manifold is said to be Einstein.

If a normal paracontact manifold $M^n(\varphi,\eta,\xi,g)$ is an η-Einstein, from (11) and (14), we get $1-n=a+b$, $\tau=na+b$, that is,

$$a=1+\frac{\tau}{n-1} \quad and \quad b=-n-\frac{\tau}{n-1}.$$

Thus (14) takes form

$$S(X,Y)=g(X,Y)(1+\frac{\tau}{n-1})-(n+\frac{\tau}{n-1})\eta(X)\eta(Y). \tag{15}$$

Theorem 1. *An n-dimensional M-projectively flat normal paracontact metric manifold M^n is an Einstein manifold.*

Proof. Let us assume that normal paracontact metric manifold M^n is M-projectively flat, then from (8) and (13), we obtain

$$R(X,Y)Z = \frac{1}{2(n-1)}\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY\}.$$

Here replacing $Z=\xi$ and using (8), we obtain

$$\eta(X)Y-\eta(Y)X=\frac{1}{n-1}\{\eta(Y)X-\eta(X)Y\},$$

which implies that

$$QX=(1-n)X,$$

or

$$S(X,Y)=(1-n)g(X,Y), \tag{16}$$

for all $X,Y\in\Gamma(TM)$. □

Proposition 1. *If A normal paracontact metric manifold $M^n(\varphi,\eta,\xi,g)$ is Weyl-conformally flat, then it an η-Einstein manifold.*

Next, let us suppose that normal paracontact metric manifold M^n is Weyl-conformal flat, then from (12), we have

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}\Big\{g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X \\ &- S(X,Z)Y\Big\}-\frac{\tau}{(n-1)(n-2)}\Big\{g(Y,Z)X-g(X,Z)Y\Big\}, \end{aligned} \tag{17}$$

for any $X,Y,Z\in\Gamma(TM)$. Taking $Z=\xi$ and making use of (8) and (11), we have

$$\begin{aligned} \eta(X)Y &- \eta(Y)X=\frac{1}{n-2}\Big\{\eta(Y)QX-\eta(X)QY+(n-1)\eta(Y)X \\ &- (n-1)\eta(X)Y\Big\}-\frac{\tau}{(n-1)(n-2)}\{\eta(Y)X-\eta(X)Y\}. \end{aligned} \tag{18}$$

This implies that

$$(1+\frac{\tau}{n-1})(\eta(X)Y-\eta(Y)X)+\eta(Y)QX-\eta(X)QY=0.$$

It follows for $Y=\xi$,

$$QX=-(n+\frac{\tau}{n-1})\eta(X)\xi+(1+\frac{\tau}{n-1})X,$$

that is, the Weyl- projectively flat normal paracontact metric manifold is an η-Einstein. Thus we have

$$S(X,Y)=(1+\frac{\tau}{n-1})g(X,Y)-(n+\frac{\tau}{n-1})\eta(X)\eta(Y). \tag{19}$$

From (15) and (19), we have the following Proposition.

Proposition 2. *A normal paracontact metric manifold $M^n(\varphi,\eta,\xi,g)$ is an η-Einstein manifold if it is Weyl-projectively flat.*

3. Pseudosymmetric Normal Paracontact Metric Manifolds

In this section, we consider pseudosymmetric normal paracontact metric manifolds.

Theorem 2. *If a normal paracontact metric manifold $M^n(\varphi,\xi,\eta,g)$ is pseudosymmetric provided $L\neq -1$, then it is an η-Einstein manifold. Furthermore, it is a semisymmetric if and only if it has a constant sectional curvature* 1*.*

Proof. We suppose that n-dimensional normal paracontact metric manifold M^n is pseudosymmetric. Then from (7), we have

$$(R(X,Y)\cdot R)(U,V,Z)=LQ(g,R)(U,V,Z;X,Y),$$

for all $X,Y,Z,U,V\in\Gamma(TM)$. It follows that

$$\begin{aligned} R(X,Y)R(U,V)Z &- R(R(X,Y)U,V)Z-R(U,R(X,Y)V)Z \\ &- R(U,V)R(X,Y)Z=-L\{R((X\wedge_g Y)U,V)Z \\ &+ R(U,(X\wedge_g Y)V)Z+R(U,V)(X\wedge_g Y)Z\}. \end{aligned} \tag{20}$$

Putting $Y=Z=\xi$ in (20) and by virtue of (9), we have

$$\begin{aligned} R(X,\xi)R(U,V)\xi &- R(R(X,\xi)U,V)\xi-R(U,R(X,\xi)V)\xi \\ &- R(U,V)R(X,\xi)\xi=-L\{R(\eta(U)X-g(X,U)\xi,V)\xi \\ &+ R(U,\eta(V)X-g(X,V)\xi)\xi+R(U,V)(X-\eta(X)\xi)\}. \end{aligned}$$

after necessary arrangements are made, we conclude

$$\begin{aligned} R(U,V)X+g(X,V)U &- g(X,U)V=L\{g(X,U)V-g(X,V)U \\ &+ g(X,V)\eta(U)\xi-g(X,U)\eta(V)\xi-R(U,V)X\}. \end{aligned}$$

if both sides of this expression are multiplied by W, we have

$$g(R(U,V)X,W) + g(X,V)g(U,W)-g(X,U)g(V,W)$$

$$
\begin{aligned}
&= L\{g(X,U)g(V,W) - g(X,V)g(U,W) \\
&+ g(X,V)\eta(U)\eta(W) - g(X,U)\eta(V)\eta(W) \\
&- g(R(U,V)X,W)\}, \qquad (21)
\end{aligned}
$$

for all $W \in \Gamma(TM)$. Here replacing $X = V = e_1, e_2, ..., e_{n-1}, e_n = \xi$ in (21) for the orthonormal basis of $\Gamma(TM)$ and by means of Ricci tensor, we get

$$
\begin{aligned}
S(U,W) &+ (n-1)g(U,W) = L\{(1-n)g(U,W) \\
&+ (n-1)\eta(U)\eta(W) - S(U,W)\},
\end{aligned}
$$

After the necessary arrangements are made, we conclude

$$
S(U,Z) + (n-1)g(U,Z) = L\Big\{(1-n)g(U,W) + (n-1)\eta(U)\eta(W) - S(U,W)\Big\},
$$

that is,

$$
S(U,W) = (1-n)g(U,W) + (n-1)\frac{L}{L+1}\eta(U)\eta(W). \qquad (22)
$$

If it is a semisymmetric, then $L = 0$ and (21) takes form

$$
R(U,V)X = g(X,U)V - g(X,V)U.
$$

This tells us that M has a constant sectional curvature 1. Conversely, if it has a constant sectional curvature 1, then we have

$$
\begin{aligned}
(R(X,Y)R)(U,V,Z) &= R(X,Y)R(U,V)Z - R(R(X,Y)U,V)Z - R(U,R(X,Y)V)Z \\
&- R(U,V)R(X,Y)Z \\
&= R(X,Y)\{g(U,Z)V - g(V,Z)U\} - R(g(X,U)Y - g(Y,U)X,V)Z \\
&- R(U,g(X,V)Y - g(Y,V)X)Z - R(U,V)\{g(X,Z)Y - g(Y,Z)X\} \\
&= g(Z,U)\{g(X,V)Y - g(Y,V)X\} - g(V,Z)\{g(X,U)Y - g(Y,U)X\} \\
&- g(X,U)\{g(Y,Z)V - g(V,Z)Y\} + g(Y,U)\{g(X,Z)V - g(V,Z)X\} \\
&- g(X,V)\{g(U,Z)Y - g(Y,Z)U\} + g(Y,V)\{g(U,Z)X - g(X,Z)U\} \\
&- g(X,Z)\{g(U,Y)V - g(Y,V)U\} + g(Y,Z)\{g(U,X)V - g(V,X)U\} \\
&= 0.
\end{aligned}
$$

This completes the proof. □

Now, we will calculate $M(X,Y)\xi$ for later use. From (8)- (11), we obtain

$$
\mathcal{M}(X,Y)\xi = \frac{1}{2}\{\eta(X)Y - \eta(Y)X\} + \frac{1}{2(n-1)}\{\eta(X)QY - \eta(Y)QX\}, \qquad (23)
$$

$$
\mathcal{M}(\xi,Y)Z = \frac{1}{2}\{\eta(Z)Y - g(Y,Z)\xi\} - \frac{1}{2(n-1)}\{S(Y,Z)\xi - \eta(Z)QY\} \qquad (24)
$$

and

$$\begin{aligned}\eta(\mathcal{M}(X,Y)Z) &= \frac{1}{2(n-1)}\{\eta(Y)S(X,Z)-\eta(X)S(Y,Z)\}\\ &+ \frac{1}{2}\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\}, \qquad (25)\end{aligned}$$

$$\begin{aligned}\mathcal{M}(\xi,X)Y &= \frac{1}{2}\{\eta(Y)X-g(X,Y)\xi\}+\frac{1}{2(n-1)}\{\eta(Y)QX\\ &- S(X,Y)\xi\}. \qquad (26)\end{aligned}$$

Theorem 3. *A normal paracontact metric manifold M^n satisfying $\mathcal{M}\cdot R=0$ is an Einstein manifold.*

Proof. We suppose that $(\mathcal{M}(X,Y)\cdot R)(U,V,Z)=0$, for any $X,Y,Z,U,V\in\Gamma(TM)$. This implies that

$$\begin{aligned}\mathcal{M}(X,Y)R(U,V)Z &- R(\mathcal{M}(X,Y)U,V)Z-R(U,M(X,Y)V)Z\\ &- R(U,V)\mathcal{M}(X,Y)Z=0. \qquad (27)\end{aligned}$$

Putting $Y=Z=\xi$ in (27), we obtain

$$\begin{aligned}\mathcal{M}(X,\xi)R(U,V)\xi &- R(\mathcal{M}(X,\xi)U,V)\xi-R(U,\mathcal{M}(X,\xi)V)\xi\\ &- R(U,V)\mathcal{M}(X,\xi)\xi=0.\end{aligned}$$

By using (9) and (24), we conclude

$$\begin{aligned}\frac{1}{2}g(X,V)U &+ \frac{1}{2(n-1)}S(X,V)U+\frac{1}{2}R(U,V)X\\ &+ \frac{1}{2(n-1)}R(U,V)QX=0. \qquad (28)\end{aligned}$$

Taking the inner product with ξ, we reach

$$\begin{aligned}\eta(U)S(X,V) &+ (n-1)\eta(U)g(X,V)+(n-1)\{\eta(V)g(X,U)\\ &- \eta(U)g(X,V)\}+\eta(V)S(X,U)-\eta(U)S(X,V)\\ &= 0,\end{aligned}$$

that is,

$$S(X,U)=(1-n)g(X,U).$$

This proves our assertion. □

Definition 1. *A semi-Riemannian manifold (M,g) is said to be the mathcalMprojective pseudosymmetric if there exists a function L on M such that*

$$R\cdot\mathcal{M}=LQ(g,\mathcal{M}),$$

where R and $\mathcal{M}$ denote the Riemannian and $\mathcal{M}$- projectively curvature tensors of M. If $L=0$, it also called the $\mathcal{M}$-projectively semisymmetric.

Theorem 4. *An $\mathcal{M}$-projective pseudosymmetric normal paracontact manifold $M^n(\varphi,\eta,\xi,g)$ is an η-Einstein manifold.*

Proof. Let us take $\mathcal{M}$-projective pseudosymmetric normal paracontact manifold $M^n(\varphi,\eta,\xi,g)$. From (5), (6), we have

$$\begin{aligned} -L\Big\{M((X\wedge_g Y)U,V)Z &+ M(U,(X\wedge_g Y)V)Z+M(U,V)(X\wedge_g Y)Z\Big\} \\ &= R(X,Y)M(U,V)Z-M(R(X,Y)U,V)Z \\ &- M(U,R(X,Y)V)Z-M(U,V)R(X,Y)Z, \quad (29) \end{aligned}$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. Setting $X=Z=\xi$ in (29), by using (23)-(26), we have

$$\begin{aligned} &- L\{\frac{1}{2}[g(Y,U)V-g(Y,V)U+g(V,Y)\eta(U)\xi-g(Y,U)\eta(V)\xi] \\ &+ \frac{1}{2(n-1)}[g(Y,U)QV-g(Y,V)QU+g(Y,V)\eta(U)Q\xi \\ &- g(Y,U)\eta(V)Q\xi]-\mathcal{M}(U,V)Y\}=\frac{1}{2}[g(Y,U)V-g(Y,V)U] \\ &+ \frac{1}{2(n-1)}[\eta(V)S(Y,U)\xi-\eta(U)S(V,Y)\xi+g(Y,U)QV \\ &- \eta(V)g(Y,U)Q\xi+\eta(U)g(Y,V)Q\xi-g(Y,V)QU] \\ &- \mathcal{M}(U,V)Y. \end{aligned}$$

If both sides of this equality are multiplied by W and by means of definition of the Ricci tensor, we obtain

$$\begin{aligned} &- L\Big\{\frac{1}{2}[g(Y,U)g(V,W)-g(Y,V)g(U,W)+g(V,Y)\eta(U)\eta(W) \\ &- g(Y,U)\eta(V)\eta(W)]+\frac{1}{2(n-1)}[g(Y,U)S(V,W)-g(Y,V)S(U,W) \\ &+ g(Y,V)\eta(U)S(\xi,W)-g(Y,U)\eta(V)S(\xi,W)]-g(\mathcal{M}(U,V)Y,W)\Big\} \\ &= \frac{1}{2}[g(Y,U)g(V,W)-g(Y,V)g(U,W)] \\ &+ \frac{1}{2(n-1)}\Big[\eta(V)S(Y,U)\eta(W)-\eta(U)S(V,Y)\eta(W) \\ &+ g(Y,U)S(V,W)-\eta(V)g(Y,U)S(\xi,W)+\eta(U)g(Y,V)S(\xi,W) \\ &- g(Y,V)S(U,W)\Big]-g(\mathcal{M}(U,V)Y,W). \end{aligned}$$

Here taking trace boht of sides for $Y=V=e_i$, for $1 \le i \le n$, in the last equality,

$$\begin{aligned}
-L\sum_{i=1}^{n}\Big\{\frac{1}{2}[\epsilon_i g(e_i,U)g(e_i,W) &- \epsilon_i g(e_i,e_i)g(U,W)+\epsilon_i g(e_i,e_i)\eta(U)\eta(W) \\
-\epsilon_i g(e_i,U)\eta(e_i)\eta(W)] &+ \frac{1}{2(n-1)}[\epsilon_i g(e_i,U)S(e_i,W)-\epsilon_i g(e_i,e_i)S(U,W) \\
&+ \epsilon_i g(e_i,e_i)\eta(U)S(\xi,W)-\epsilon_i g(e_i,U)\eta(e_i)S(\xi,W)] \\
&- \epsilon_i g(M(U,e_i)e_i,W)\Big\} \\
&= \sum_{i=1}^{n}\epsilon_i\Big\{\frac{1}{2}[\epsilon_i g(e_i,U)g(e_i,W)-\epsilon_i g(e_i,e_i)g(U,W)] \\
&+ \frac{1}{2(n-1)}[\epsilon_i\eta(e_i)S(e_i,U)\eta(W)-\epsilon_i\eta(U)S(e_i,e_i)\eta(W) \\
&+ \epsilon_i g(e_i,U)S(e_i,W)-\epsilon_i\eta(e_i)g(e_i,U)S(\xi,W) \\
&+ \epsilon_i\eta(U)g(e_i,e_i)S(\xi,W)-\epsilon_i g(e_i,e_i)S(U,W)] \\
&- \epsilon_i g(M(U,e_i)e_i,W)\Big\}, \qquad (30)
\end{aligned}$$

where ϵ_i is the signature $\{e_i\}$. On the other hand, by direct calculations, we have

$$\epsilon_i g(M(U,e_i)e_i,W)=\frac{1}{2(n-1)}\{n.S(U,W)-\tau.g(U,W)\}.$$

Making use of (30) and after the necessary arrangements are revised, we get

$$S(U,W)=\frac{(1-n)(n-1)+\tau}{2n-1}g(U,W)+\frac{n(1-n)-\tau}{(2n-1)(1+L)}\eta(U)\eta(W),$$

which proves the theorem. □

Definition 2. *A normal paracontact manifold $M^n(\varphi,\eta,\xi,g)$ is said to be the Weyl-pseudosymmetric if there exists a function L on M such that*

$$R\cdot C=LQ(g,C),$$

where R and C denote the Riemannian and Weyl-conformal curvature tensors of M. If $L=0$, then it also called the Weyl-semisymmetric.

Now, we consider the Weyl-conformal curvature tensor of M^n given by (12) for later use.

$$\begin{aligned}
C(X,Y)\xi &= \left(\frac{1-n-\tau}{(n-1)(n-2)}\right)(\eta(X)Y-\eta(Y)X) \\
&+ \frac{1}{n-2}(\eta(X)QY-\eta(Y)QX) \qquad (31)
\end{aligned}$$

and

$$\begin{aligned} C(\xi, X)Y &= \left(\frac{1-n-\tau}{(n-1)(n-2)}\right)(\eta(Y)X - g(X,Y)\xi) \\ &+ \frac{1}{n-2}(\eta(Y)QX - S(X,Y)\xi). \end{aligned} \tag{32}$$

Theorem 5. *The Weyl-pseudosymmetric normal paracontact metric manifold $M^n(\varphi, \eta, \xi, g)$ is an η-Einstein manifold.*

Proof. Let $M^n(\varphi, \eta, \xi, g)$ be the Weyl-pseudosymmetric, then there is a function L such that

$$(R(X,Y)\cdot C)(U,V,Z) = LQ(g,C)(U,V,Z;X,Y),$$

for all $X, Y, U, V, Z \in \Gamma(TM)$. This implies that

$$\begin{aligned} R(X,Y)C(U,V)Z &- C(R(X,Y)U,V)Z - C(U,R(X,Y)V)Z \\ &- C(U,V)R(X,Y)Z = -L\Big\{C((X\wedge_g Y)U,V)Z \\ &+ C(U,(X\wedge_g Y)V)Z + C(U,V)(X\wedge_g Y)Z\Big\}. \end{aligned} \tag{33}$$

Here setting $X = Z = \xi$ in (33), we have

$$\begin{aligned} R(\xi,Y)C(U,V)\xi &- C(R(\xi,Y)U,V)\xi - C(U,R(\xi,Y)V)\xi \\ &- C(U,V)R(\xi,Y)\xi = -L\Big\{C((\xi\wedge_g Y)U,V)\xi \\ &+ C(U,(\xi\wedge_g Y)V)\xi + C(U,V)(\xi\wedge_g Y)\xi\Big\}. \end{aligned} \tag{34}$$

After the necessary calculations, we reach at

$$\begin{aligned} &\frac{1-n-\tau}{(n-1)(n-2)}\{g(Y,U)V - g(Y,V)U\} \\ +\ &\frac{1}{n-2}\{\eta(V)S(Y,U)\xi - \eta(U)S(V,Y)\xi \\ +\ &g(Y,U)QV - \eta(V)g(Y,U)Q\xi \\ +\ &\eta(U)g(Y,V)Q\xi - g(Y,V)QU\} - C(U,V)Y \\ =\ &-L\{\frac{1-n-\tau}{(n-1)(n-2)}(g(Y,U)V - g(Y,V)U - g(Y,U)\eta(V)\xi \\ +\ &g(Y,V)\eta(U)\xi) + \frac{1}{n-2}(g(Y,U)QV - g(Y,V)QU \\ -\ &\eta(V)g(Y,U)Q\xi + \eta(U)g(Y,V)Q\xi) - C(U,V)Y\}. \end{aligned} \tag{35}$$

If both sides of the equality are multiplied by W, we obtain

$$\frac{1-n-\tau}{(n-1)(n-2)}\{g(Y,U)g(V,W) - g(Y,V)g(U,W)\}$$

$$
\begin{aligned}
&+ \quad \frac{1}{n-2}\{\eta(V)S(Y,U)\eta(W) - \eta(U)S(V,Y)\eta(W) \\
&+ \quad g(Y,U)S(V,W) - \eta(V)g(Y,U)S(\xi,W) \\
&+ \quad \eta(U)g(Y,V)S(\xi,W) - g(Y,V)S(W,U)\} \\
&- \quad g(C(U,V)Y,W) \\
&= \quad -L\{\frac{1-n-\tau}{(n-1)(n-2)}(g(Y,U)g(V,W) - g(Y,V)g(U,W) \\
&- \quad g(Y,U)\eta(V)\eta(W) + g(Y,V)\eta(U)\eta(W)) \\
&+ \quad \frac{1}{n-2}(g(Y,U)S(V,W) - g(Y,V)S(U,W) \\
&- \quad \eta(V)g(Y,U)S(\xi,W) + \eta(U)g(Y,V)S(\xi,W)) \\
&- \quad g(C(U,V)Y,W)\}. \qquad (36)
\end{aligned}
$$

Putting $Y = V = e_1, e_2, ...e_{n-1}, e_n = \xi$ in (36) for the orthonormal basis of $\Gamma(TM)$ and taking into account definition of Ricci tensor, we have

$$
\begin{aligned}
& \quad \frac{1-n-\tau}{(n-1)(n-2)}\sum_{i=1}^{n}\Big\{\epsilon_i\{g(e_i,U)g(e_i,W) - \epsilon_i g(e_i,e_i)g(U,W)\} \\
&+ \quad \frac{1}{n-2}\{\epsilon_i\eta(e_i)S(e_i,U)\eta(W) - \epsilon_i\eta(U)S(e_i,e_i)\eta(W) \\
&+ \quad \epsilon_i g(e_i,U)S(e_i,W) - \epsilon_i\eta(e_i)g(e_i,U)S(\xi,W) \\
&+ \quad \epsilon_i\eta(U)g(e_i,e_i)S(\xi,W) - \epsilon_i g(e_i,e_i)S(W,U)\} \\
&- \quad \epsilon_i g(C(U,e_i)e_i,W)\Big\} \\
&= \quad -L\{\frac{1-n-\tau}{(n-1)(n-2)}\sum_{i=1}^{n}\Big\{\epsilon_i(g(e_i,U)g(e_i,W) - \epsilon_i g(e_i,e_i)g(U,W) \\
&- \quad \epsilon_i g(e_i,U)\eta(e_i)\eta(W) + \epsilon_i g(e_i,e_i)\eta(U)\eta(W)) \\
&+ \quad \frac{1}{n-2}(\epsilon_i g(e_i,U)S(e_i,W) - \epsilon_i g(e_i,e_i)S(U,W) \\
&- \quad \epsilon_i\eta(e_i)g(e_i,U)S(\xi,W) + \epsilon_i\eta(U)g(e_i,e_i)S(\xi,W)) \\
&- \quad \epsilon_i g(C(U,e_i)e_i,W)\}\Big\}. \qquad (37)
\end{aligned}
$$

By using (11) and after the necessary abbreviations, (37) implies that

$$
S(U,W) = (1 - \frac{\tau}{n-1})g(U,W) - (n + \frac{\tau}{n-1})\eta(U)\eta(W).
$$

This proves our assertion. □

Now, we will give an non-trivial example for illustration our results.

Example 1. *Let us the 5-dimensional manifold*

$$M^5 = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in R,\}$$

where (x_i) *denote the cartesian coordinate in* $\mathbb{R}^5$ *for* $1 \leq i \leq 5$. *Then the vector fields*

$$e_1 = \frac{\partial}{\partial x_1}, e_2 = \frac{\partial}{\partial x_2}, e_3 = 2x_2\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, e_4 = 2x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}, e_5 = -2x_4\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_4}$$

are linearly independent at each point of M^5. *By* g, *we denote the semi-Riemannian metric tensor such that*

$$g(e_i, e_j) = \begin{cases} 1; & i = j = 1, 3, 4 \\ -1; & i = j = 2, 5 \\ 0; & i \neq j \end{cases}$$

Let η *be the 1-form defined by* $\eta(X) = g(X, e_1)$ *for all* $X \in \Gamma(TM)$. *Now, we definite the paracontact metric structure* φ *such that*

$$\varphi e_1 = 0, \quad \varphi e_2 = -e_3, \quad \varphi e_3 = -e_2, \quad \varphi e_4 = -e_5, \quad \varphi e_5 = -e_4.$$

Then we can easily see that

$$\eta(e_5) = 1, \quad \varphi^2 X = X - \eta(X)\xi, \quad e_5 = \xi$$

and

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \Gamma(\widetilde{M})$. *Thus* $M^5(\varphi, \eta, \xi, g)$ *defines an almost paracontact metric manifold. By* $\widetilde{\nabla}$, *we denote the Levi-Civita connection on* $\widetilde{M}$. *Then by direct calculations, we have non-zero the components*

$$[e_2, e_3] = 2e_1, \quad [e_3, e_4] = 2e_1, \quad [e_4, e_5] = -2e_1.$$

Let ∇ *be the Levi-Civita connection on* M. *Using the properties of paracontact metric structure and Kozsul formulae, we can observe the non-zero components*

$$\nabla e_2 e_1 = -e_3 = \varphi e_2, \quad \nabla e_3 e_1 = -e_2 = \varphi e_3, \quad \nabla e_4 e_1 = -e_5 = \varphi e_4, \quad \nabla e_5 e_1 = -e_4 = \varphi e_5$$

Thus one can easily verified

$$\widetilde{\nabla}_X e_1 = \varphi X,$$

for all $X \in \Gamma(TM)$ *This tells us that* $M^5(\varphi, \eta, \xi, g)$ *is a normal paracontact metric manifold with paracontact metric structure* (φ, η, ξ, g). *By straightforward calculations, we can easily see that non-zero components of the Riemannian curvature tensor* R,

$$R(e_i, e_1)e_1 = -e_i, \quad 2 \leq i \leq 5.$$

This tell us that

$$R(X,Y)Z = g(X,Z)Y - g(Y,Z)X,$$

for all $X, Y, Z \in \Gamma(TM)$*, that is,* $\widetilde{M}(\varphi, \eta, \xi, g)$ *is real space form with constant sectional curvature* 1*.*

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