Celal Bayar University Journal of Science

Auto-Bäcklund Transformation for Travelling Wave Solutions of Some Nonlinear Partial Differential Equations

İbrahim Enam İnan[1*], Ünal İç[1]
[1]Firat University, Faculty of Education, 23119 Elazig, TURKEY
*einan@firat.edu.tr
*Orcid: 0000-0003-3681-0497

Received: 03 September 2019
Accepted: 15 June 2020
DOI: 10.18466/cbayarfbe.614476

Abstract

In this paper, we implemented Auto-B$\ddot{a}$cklund transformation for finding the travelling wave solutions of the complexly coupled KdV equations and the sixth order equation of the Burgers hierarchy. These solutions are hyperbolic function solutions and exponential function solutions. It was observed using the Mathematica program that these solutions provided the nonlinear partial differential equation and the nonlinear partial differential equation pair. Then, the solutions of these equations are compared with the solutions of the same equations obtained in the literature using other methods. As a result of the comparison, it was seen that some solutions are the same and some solutions are similar. The Auto-B$\ddot{a}$cklund transformation used in this article is a powerful method for finding traveling wave solutions of nonlinear partial differential equations.

Keywords: Auto-Bäcklund transformation, complexly coupled KdV equations, sixth order equation of the Burgers hierarchy, travelling wave solutions.

1.Introduction

Nonlinear partial differential equations (NPDEs) plays an important role in applied sciences. Some analytical methods for solving these equations exist in the literature [1-8]. Along with these methods, there are several methods of solving such equations by using an auxiliary equation. By using such methods, partial differential equations are converted to ordinary differential equations and the solutions of partial differential equations are found with the help of these ordinary differential equations. Some of these methods and their applications are given in [9-30].
We used the Auto- Bäcklund transformation for finding the travelling wave solutions of the complexly coupled KdV equations and the sixth order equation of the Burgers hierarchy. This method is presented in [7]. J. Liu et al. [28] obtained the exact solutions of complexly coupled KdV equations by using Jacobi elliptic function method. I.E. Inan and D. Kaya [25] reached exact solutions of complexly coupled KdV equations by means of using Generalized tanh function method. A.M. Wazwaz [17] achieved multiple kink solutions and multiple singular kink solutions of Burgers hierarchy equations through using Hirota bilinear method. I.E. Inan et al. [19] obtained solitary wave solutions of third order equation of Burgers hierarchy (Sharma-Tasso-Olver equation) and solitary wave solutions of fourth order equation of Burgers hierarchy by using Auto-Bäcklund transformation.

2. Analysis of Method

Let's introduce the method briefly. Consider a general partial differential equation of two variables,

$$Q(u, u_t, u_x, u_{xx}, \dots) = 0. \quad \textbf{(2.1)}$$

We can express the solution of Eq.(1) as below,

$$u = \frac{\partial^M}{\partial x^M} f(w) + u_0, \quad \textbf{(2.2)}$$

where M is a positive integer is found as the result of balancing the highest order linear term and the highest order nonlinear term found in the equation. $f = f(w)$ and $w = w(x,t)$ are unknown functions, u and u_0 are two solutions of Eq.(2.1). Substituting (2.2) into (2.1), putting all terms of the highest degree of w_x together.

And setting its coefficients to zero leads to an ordinary equation, from which $f(w)$ is obtained. Replace the nonlinear term of various derivatives of $f(w)$ in expression by the corresponding higher order derivatives of $f(w)$. Collecting all terms with $f', f'', \ldots$, and setting their coefficients to zero respectively, we obtain a set of equations for $w(x,t)$ from which the compatability conditions on w will be obtained [7].

3. Applications

3.1 Application

We consider complexly coupled KdV equations [28]

$$u_t - 6uu_x - 6vv_x - u_{3x} = 0,$$
$$v_t - 6uv_x - 6vu_x - v_{3x} = 0. \quad \textbf{(3.1)}$$

In accordance with the idea of improved HB [7]. We investigate for Auto-Bäcklund transformation of Eq.(3.1). When balancing uu_x and vv_x with u_{xxx} or uv_x and vu with v_{xxx} then gives $M_1 = 2$ and $M_2 = 2$. Hence, we can write

$$u = \frac{\partial^2}{\partial x^2} f(w) + u_0 = f'' w_x^2 + f' w_{2x} + u_0,$$

$$v = \frac{\partial^2}{\partial x^2} g(w) + v_0 = g'' w_x^2 + g' w_{2x} + v_0. \quad \textbf{(3.2)}$$

where $f = f(w)$, $g = g(w)$, $w = w(x,t)$, $u_0 = u_0(x,t)$ and $v_0 = v_0(x,t)$. Here $f = f(w)$, $g = g(w)$ and $w = w(x,t)$ are undetermined functions u, u_0, v and v_0 are four solutions of Eq.(3.1). Substituting (3.2) into Eq.(3.1), we obtain

$$u_t = f''' w_t w_x^2 + 2f'' w_x w_{xt} + f'' w_t w_{2x} + f' w_{2xt} + (u_0)_t$$
$$-6uu_x = -6f'' f''' w_x^5 - 18(f'')^2 w_x^3 w_{2x} - 6f' f'' w_x^2 w_{3x} - 6f'' w_x^2 (u_0)_x - 6f' f''' w_x^3 w_{2x} - 18f' f'' w_x w_{2x}^2 - 6(f')^2 w_{2x} w_{3x} - 6f' w_{2x}(u_0)_x - 6f''' w_x^3 u_0 - 18f'' w_x w_{2x} u_0 - 6f' w_{3x} u_0 - 6u_0(u_0)_x$$
$$-6vv_x = 6g'' g''' w_x^5 - 18(g'')^2 w_x^3 w_{2x} - 6g' g'' w_x^2 w_{3x} - 6g'' w_x^2 (v_0)_x - 6g' g''' w_x^3 w_{2x} - 18g' g'' w_x w_{2x}^2 - 6(g')^2 w_{2x} w_{3x} - 6g' w_{2x}(v_0)_x - 6g''' w_x^3 v_0 - 18g'' w_x w_{2x} v_0 - 6g' w_{3x} v_0 - 6v_0(v_0)_x$$
$$-u_{3x} = -f^{(5)} w_x^5 - 10f^{(4)} w_x^3 w_{2x} - 15f''' w_x w_{2x}^2 - 10f''' w_x^2 w_{3x} - 10f'' w_{2x} w_{3x} - 5f'' w_x w_{4x} - f' w_{5x} - (u_0)_{3x} = 0.$$

and

$$v_t = g''' w_t w_x^2 + 2g'' w_x w_{xt} + g'' w_t w_{2x} + g' w_{2xt} + (v_0)_t$$
$$-6uv_x = -6f'' g''' w_x^5 - 18f'' g'' w_x^3 w_{2x} - 6f'' g' w_x^2 w_{3x} - 6f'' w_x^2 (v_0)_x - 6f' g''' w_x^3 w_{2x} - 18f' g'' w_x w_{2x}^2 - 6f' g' w_{2x} w_{3x} - 6f' w_{2x}(v_0)_x - 6g''' w_x^3 u_0 - 18g'' w_x w_{2x} u_0 - 6g' w_{3x} u_0 - 6u_0(v_0)_x$$
$$-6vu_x = -6g'' f''' w_x^5 - 18g'' f'' w_x^3 w_{2x} - 6g'' f' w_x^2 w_{3x} - 6g'' w_x^2 (u_0)_x - 6g' f''' w_x^3 w_{2x} - 18g' f'' w_x w_{2x}^2 - 6g' f' w_{2x} w_{3x} - 6g' w_{2x}(u_0)_x - 6f''' w_x^3 v_0 - 18f'' w_x w_{2x} v_0 - 6f' w_{3x} v_0 - 6v_0(u_0)_x$$
$$-v_{3x} = -g^{(5)} w_x^5 - 10g^{(4)} w_x^3 w_{2x} - 15g''' w_x w_{2x}^2 - 10g''' w_x^2 w_{3x} - 10g'' w_{2x} w_{3x} - 5g'' w_x w_{4x} - g' w_{5x} - (v_0)_{3x} = 0.$$

and

$$\left(-6f'' f''' - 6g'' g''' - f^{(5)}\right) w_x^5 + \left(-18(f'')^2 w_x^3 w_{2x} - 6f' f''' w_x^3 w_{2x} - 18(g'')^2 w_x^3 w_{2x} - 6g' g''' w_x^3 w_{2x} - 10f^{(4)} w_x^3 w_{2x}\right) + \left(f''' w_t w_x^2 - 6f' f'' w_x^2 w_{3x} - 18f' f'' w_x w_{2x}^2 - 6f''' w_x^3 u_0 - 6g' g'' w_x^2 w_{3x} - 18g' g'' w_x w_{2x}^2 - 6g''' w_x^3 v_0 - 15f''' w_x w_{2x}^2 - 10f''' w_x^2 w_{3x}\right) + \left(2f'' w_x w_{xt} + f'' w_t w_{2x} - 6f'' w_x^2 (u_0)_x - 6(f')^2 w_{2x} w_{3x} - 18f'' w_x w_{2x} u_0 - 6g'' w_x^2 (v_0)_x - 6(g')^2 w_{2x} w_{3x} - 18g'' w_x w_{2x} v_0 - 10f'' w_{2x} w_{3x} - 5f'' w_x w_{4x}\right) + \left(f' w_{2xt} - 6f' w_{2x}(u_0)_x - 6g' w_{2x}(v_0)_x - 6f' w_{3x} u_0 - 6g' w_{3x} v_0 - f' w_{5x}\right) = 0. \quad \textbf{(3.3)}$$

$$\left(-6f'' g''' - 6g'' f''' - g^{(5)}\right) w_x^5 + \left(-18f'' g'' w_x^3 w_{2x} - 6f' g''' w_x^3 w_{2x} - 18g'' f'' w_x^3 w_{2x} - 6g' f''' w_x^3 w_{2x} - 10g^{(4)} w_x^3 w_{2x}\right) + \left(g''' w_t w_x^2 - 6f'' g' w_x^2 w_{3x} - 18f' g'' w_x w_{2x}^2 - 6g''' w_x^3 u_0 - 6g'' f' w_x^2 w_{3x} - 18g' f'' w_x w_{2x}^2 - 6f''' w_x^3 v_0 - 15g''' w_x w_{2x}^2 - 10g''' w_x^2 w_{3x}\right) + \left(2g'' w_x w_{xt} + g'' w_t w_{2x} - 6f'' w_x^2 (v_0)_x - 6f' g' w_{2x} w_{3x} - 18g'' w_x w_{2x} u_0 - 6g'' w_x^2 (u_0)_x - 6g' f' w_{2x} w_{3x} - 18f'' w_x w_{2x} v_0 - 10g'' w_{2x} w_{3x} - 5g'' w_x w_{4x}\right) + \left(g' w_{2xt} - 6f' w_{2x}(v_0)_x - 6g' w_{2x}(u_0)_x - 6f' w_{3x} v_0 - 6g' w_{3x} u_0 - g' w_{5x}\right) = 0. \quad \textbf{(3.4)}$$

Setting the coefficients of w_x^5 in (3.3) and (3.4) to zero respectively, we obtain a set of ordinary differential equations

$$-6f'' f''' - 6g'' g''' - f^{(5)} = 0,$$
$$-6f'' g''' - 6g'' f''' - g^{(5)} = 0,$$

which have solutions

$$f = g = lnw, \quad \textbf{(3.5)}$$

there by from (7) it holds that

$$f' = g',$$
$$f' g' = (f')^2 = -f'' = (g')^2,\ f'' = g'',$$
$$f' f'' = -\tfrac{1}{2} f''' = g' g'' = f'' g' = f' g'',\ f''' = g''',$$
$$(f'')^2 = -\tfrac{1}{6} f^{(4)} = (g'')^2 = f'' g'', f' f''' = -\tfrac{1}{3} f^{(4)} = g' g''' = f' g''' = g' f''', f^{(4)} = g^{(4)}. \quad \textbf{(3.6)}$$

Celal Bayar University Journal of Science
Volume 16, Issue 2, 2020, p 229-236
Doi: 10.18466/cbayarfbe.614476

By using (3.3), (3.4) and (3.6) can be written as the sum of some terms with f' and f'' setting their coefficients to zero will lead to

$$w_x(w_tw_x - 4w_xw_{3x} + 3w_{2x}^2 - 6w_x^2u_0 - 6w_x^2v_0) = 0,$$
$$\frac{\partial}{\partial x}(w_tw_x - 4w_xw_{3x} + 3w_{2x}^2 - 6w_x^2u_0 - 6w_x^2v_0) +$$
$$w_x(w_{xt} - 6w_{2x}u_0 - 6w_{2x}v_0 - w_{4x}) = 0,$$
$$\frac{\partial}{\partial x}(w_{xt} - 6w_{2x}u_0 - 6w_{2x}v_0 - w_{4x}) = 0.$$

and

$$w_x(w_tw_x - 4w_xw_{3x} + 3w_{2x}^2 - 6w_x^2u_0 - 6w_x^2v_0) = 0,$$
$$\frac{\partial}{\partial x}(w_tw_x - 4w_xw_{3x} + 3w_{2x}^2 - 6w_x^2u_0 - 6w_x^2v_0) +$$
$$w_x(w_{xt} - 6w_{2x}u_0 - 6w_{2x}v_0 - w_{4x}) = 0,$$
$$\frac{\partial}{\partial x}(w_{xt} - 6w_{2x}u_0 - 6w_{2x}v_0 - w_{4x}) = 0.$$

Above equations are satisfied provided that

$$w_tw_x - 4w_xw_{3x} + 3w_{2x}^2 - 6w_x^2u_0 - 6w_x^2v_0 = 0, \quad \textbf{(3.7)}$$

$$w_{xt} - 6w_{2x}u_0 - 6w_{2x}v_0 - w_{4x} = 0. \quad \textbf{(3.8)}$$

From (3.2) and (3.5), we obtain $B\ddot{a}cklund$ transformation of Eq.(2.1)

$$u = \frac{\partial^2}{\partial x^2}lnw + u_0,\ v = \frac{\partial^2}{\partial x^2}lnw + v_0, \quad \textbf{(3.9)}$$

where w satisfying (3.7) and (3.8). We take initial solutions of Eq.(2.1) as $u_0 = v_0 = 0$,
then (3.7),(3.8) and (3.9) respectively reduce to

$$w_tw_x - 4w_xw_{3x} + 3w_{2x}^2 = 0, \quad \textbf{(3.10)}$$

$$w_{xt} - w_{4x} = 0. \quad \textbf{(3.11)}$$

$$u = \frac{\partial^2}{\partial x^2}lnw,\ v = \frac{\partial^2}{\partial x^2}lnw. \quad \textbf{(3.12)}$$

Specially, we take a solution of (3.10) and (3.11)

$$w = 1 + e^{c(x+c^2t)}. \quad \textbf{(3.13)}$$

Then, solutions of Eq.(3.1) can be written by using Eq.(3.12) as following

$$u(x,t) = v_1(x,t) = \frac{c^2e^{c(x+c^2t)}}{\left(1+e^{c(x+c^2t)}\right)^2}, \quad \textbf{(3.14)}$$

or

$$u(x,t) = v_2(x,t) = \frac{c^2}{4}Sech^2[\frac{c}{2}(x + c^2t)]. \quad \textbf{(3.15)}$$

where c is arbitrary constant.

3.2 Application

The sixth order equation of the Burgers hierarchy is as follows [17]

$$u_t + u_{6x} + 21u_xu_{4x} + 35u_{2x}u_{3x} + 6uu_{5x} +$$
$$15u^2u_{4x} + 90uu_xu_{3x} + 60uu_{2x}^2 + 105u_{2x}u_x^2 +$$
$$20u^3u_{3x} + 150u^2u_xu_{2x} + 90uu_x^3 + 15u^4u_{2x} +$$
$$60u^3u_x^2 + 6u^5u_x = 0. \quad \textbf{(3.16)}$$

In accordance with the idea of improved HB [7]. We investigate for $B\ddot{a}cklund$ transformation of Eq.(3.16). When balancing $u_{2x}u_{3x}$ with u_{6x} then gives $M = 1$. Hence, we can write

$$u = \frac{\partial}{\partial x}f(w) + u_0 = f'w_x + u_0, \quad \textbf{(3.17)}$$

where $f = f(w)$, $w = w(x,t)$, $u_0 = u_0(x,t)$. Here $f = f(w)$ and $w = w(x,t)$ are undetermined functions, u and u_0 are two solutions of Eq.(3.16). Substituting (3.17) into Eq.(3.16) we obtain

$$u_t = f''w_tw_x + f'w_{xt} + (u_0)_t$$
$$u_{6x} = f^{(7)}w_x^7 + 21f^{(6)}w_x^5w_{2x} + 105f^{(5)}w_x^3w_{2x}^2 +$$
$$35f^{(5)}w_x^4w_{3x} + 105f^{(4)}w_xw_{2x}^3 +$$
$$210f^{(4)}w_x^2w_{2x}w_{3x} + 35f^{(4)}w_x^3w_{4x} +$$
$$105f'''w_{2x}^2w_{3x} + 70f'''w_xw_{3x}^2 + 105f'''w_xw_{2x}w_{4x} +$$
$$21f'''w_x^2w_{5x} + 35f''w_{3x}w_{4x} + 21f''w_{2x}w_{5x} +$$
$$7f''w_xw_{6x} + f'w_{7x} + (u_0)_{6x}$$
$$21u_xu_{4x} = 21f^{(5)}w_x^5(u_0)_x + 210f^{(4)}w_x^3w_{2x}(u_0)_x +$$
$$21(u_0)_x(u_0)_{4x} + 21f^{(5)}f'w_x^5w_{2x} +$$
$$210f^{(4)}f'w_x^3w_{2x}^2 + 21f'w_{5x}(u_0)_x + 21f'w_{2x}(u_0)_{4x} +$$
$$21(f')^2w_{2x}w_{5x} + 21f^{(5)}f''w_x^7 + 210f^{(4)}f''w_x^5w_{2x} +$$
$$210f''w_{2x}w_{3x}(u_0)_x + 105f''w_xw_{4x}(u_0)_x +$$
$$21f''w_x^2(u_0)_{4x} + 210f'f''w_{2x}^2w_{3x} +$$
$$105f'f''w_xw_{2x}w_{4x} + 21f'f''w_x^2w_{5x} +$$
$$210(f'')^2w_x^2w_{2x}w_{3x} + 105(f'')^2w_x^3w_{4x} +$$
$$315f'''w_xw_{2x}^2(u_0)_x + 210f'''w_x^2w_{3x}(u_0)_x +$$
$$315f'f'''w_xw_{2x}^3 + 210f'f'''w_x^2w_{2x}w_{3x} +$$
$$315f''f'''w_x^3w_{2x}^2 + 210f''f'''w_x^4w_{3x}$$
$$35u_{2x}u_{3x} = 35f^{(4)}w_x^4(u_0)_{xx} + 35(u_0)_{2x}(u_0)_{3x} +$$
$$35f^{(4)}f'w_x^4w_{3x} + 35f'w_{3x}(u_0)_{3x} + 35f'w_{4x}(u_0)_{2x} +$$
$$35(f')^2w_{3x}w_{4x} + 105f^{(4)}f''w_x^5w_{2x} +$$
$$105f''w_xw_{2x}(u_0)_{3x} + 105f''w_{2x}^2(u_0)_{2x} +$$
$$140f''w_xw_{3x}(u_0)_{2x} + 105f'f''w_{2x}^2w_{3x} +$$
$$140f'f''w_xw_{3x}^2 + 105f'f''w_xw_{2x}w_{4x} +$$
$$315(f'')^2w_xw_{2x}^3 + 420(f'')^2w_x^2w_{2x}w_{3x} +$$
$$35f^{(4)}f'''w_x^7 + 35f'''w_x^3(u_0)_{3x} +$$
$$210f'''w_x^2w_{2x}(u_0)_{2x} + 210f'f'''w_x^2w_{2x}w_{3x} +$$
$$35f'f'''w_x^3w_{4x} + 735f''f'''w_x^3w_{2x}^2 +$$
$$140f''f'''w_x^4w_{3x} + 210(f''')^2w_x^5w_{2x}$$
$$6uu_{5x} = 6f^{(6)}w_x^6u_0 + 90f^{(5)}w_x^4w_{2x}u_0 +$$
$$270f^{(4)}w_x^2w_{2x}^2u_0 + 120f^{(4)}w_x^3w_{3x}u_0 + 6u_0(u_0)_{5x} +$$
$$60f''w_{3x}^2u_0 + \ +90f''w_{2x}w_{4x}u_0 + 36f''w_xw_{5x}u_0 +$$
$$6f^{(6)}f'w_x^7 + 90f^{(5)}f'w_x^5w_{2x} + 270f^{(4)}f'w_x^3w_{2x}^2 +$$

Celal Bayar University Journal of Science
Volume 16, Issue 2, 2020, p 229-236
Doi: 10.18466/cbayarfbe.614476

$$
\begin{aligned}
&120f^{(4)}f'w_x^4w_{3x} + 6f'w_{6x}u_0 + 6f'w_x(u_0)_{5x} + \\
&60f'f''w_xw_{3x}^2 + 6(f')^2w_xw_{6x} + 90f'f''w_xw_{2x}w_{4x} + \\
&36f'f''w_x^2w_{5x} + 90f'''w_{2x}^3u_0 + \\
&360f'''w_xw_{2x}w_{3x}u_0 + 90f'''w_x^2w_{4x}u_0 + \\
&90f'f'''w_xw_{2x}^3 + 360f'f'''w_x^2w_{2x}w_{3x} + \\
&90f'f'''w_x^3w_{4x}
\end{aligned}
$$

$$
\begin{aligned}
&15u^2u_{4x} = 15f^{(5)}w_x^5(u_0^2) + 150f^{(4)}w_x^3w_{2x}(u_0^2) + \\
&15(u_0^2)(u_0)_{4x} + 30f^{(5)}f'w_x^6u_0 + \\
&300f^{(4)}f'w_x^4w_{2x}u_0 + 15f'w_{5x}(u_0^2) + \\
&30f'w_xu_0(u_0)_{4x} + 15f^{(5)}(f')^2w_x^7 + \\
&150f^{(4)}(f')^2w_x^5w_{2x} + 30(f')^2w_xw_{5x}u_0 + \\
&15(f')^2w_x^2(u_0)_{4x} + 15(f')^3w_x^2w_{5x} + \\
&150f''w_{2x}w_{3x}u_0^2 + 75f''w_xw_{4x}u_0^2 + \\
&300f'f''w_xw_{2x}w_{3x}u_0 + 150f'f''w_x^2w_{4x}u_0 + \\
&150(f')^2f''w_x^2w_{2x}w_{3x} + 75(f')^2f''w_x^3w_{4x} + \\
&225f'''w_xw_{2x}^2u_0^2 + 150f'''w_x^2w_{3x}u_0^2 + \\
&450f'f'''w_x^2w_{2x}^2u_0 + 300f'f'''w_x^3w_{3x}u_0 + \\
&225(f')^2f'''w_x^3w_{2x}^2 + 150(f')^2f'''w_x^4w_{3x}
\end{aligned}
$$

$$
\begin{aligned}
&90uu_xu_{3x} = 90f^{(4)}w_x^4u_0(u_0)_x + 90u_0(u_0)_x(u_0)_{3x} + \\
&90f^{(4)}f'w_x^4w_{2x}u_0 + 90f^{(4)}f'w_x^5(u_0)_x + \\
&90f'w_{4x}u_0(u_0)_x + 90f'w_{2x}u_0(u_0)_{3x} + \\
&90f'w_x(u_0)_x(u_0)_{3x} + 90f^{(4)}(f')^2w_x^5w_{2x} + \\
&90(f')^2w_{2x}w_{4x}u_0 + 90(f')^2w_xw_{4x}(u_0)_x + \\
&90(f')^2w_xw_{2x}(u_0)_{3x} + 90(f')^3w_xw_{2x}w_{4x} + \\
&90f^{(4)}f''w_x^6u_0 + 270f''w_{2x}^2u_0(u_0)_x + \\
&360f''w_xw_{3x}u_0(u_0)_x + 90f''w_x^2u_0(u_0)_{3x} + \\
&90f^{(4)}f'f''w_x^7 + 270f'f''w_{2x}^3u_0 + \\
&360f'f''w_xw_{2x}w_{3x}u_0 + 90f'f''w_x^2w_{4x}u_0 + \\
&270f'f''w_xw_{2x}^2(u_0)_x + 360f'f''w_x^2w_{3x}(u_0)_x + \\
&90f'f''w_x^3(u_0)_{3x} + 270(f')^2f''w_xw_{2x}^3 + \\
&360(f')^2f''w_x^2w_{2x}w_{3x} + 90(f')^2f''w_x^3w_{4x} + \\
&270(f'')^2w_x^2w_{2x}^2u_0 + 360(f'')^2w_x^3w_{3x}u_0 + \\
&270f'(f'')^2w_x^3w_{2x}^2 + 360f'(f'')^2w_x^4w_{3x} + \\
&540f'''w_x^2w_{2x}u_0(u_0)_x + 540f'f'''w_x^2w_{2x}^2u_0 + \\
&540f'f'''w_x^3w_{2x}(u_0)_x + 540(f')^2f'''w_x^3w_{2x}^2 + \\
&540f''f'''w_x^4w_{2x}u_0 + 540f'f''f'''w_x^5w_{2x}
\end{aligned}
$$

$$
\begin{aligned}
&60uu_{2x}^2 = 60u_0(u_0)_{2x}^2 + 120f'w_{3x}u_0(u_0)_{2x} + \\
&60f'w_x(u_0)_{2x}^2 + 60(f')^2w_{3x}^2u_0 + \\
&120(f')^2w_xw_{3x}(u_0)_{2x} + 60(f')^3w_xw_{3x}^2 + \\
&360f''w_xw_{2x}u_0(u_0)_{2x} + 360f'f''w_xw_{2x}w_{3x}u_0 + \\
&360f'f''w_x^2w_{2x}(u_0)_{2x} + 360(f')^2f''w_x^2w_{2x}w_{3x} + \\
&540(f'')^2w_x^2w_{2x}^2u_0 + 540f'(f'')^2w_x^3w_{2x}^2 + \\
&120f'''w_x^3u_0(u_0)_{2x} + 120f'f'''w_x^3w_{3x}u_0 + \\
&120f'f'''w_x^4(u_0)_{2x} + 120(f')^2f'''w_x^4w_{3x} + \\
&360f''f'''w_x^4w_{2x}u_0 + 360f'f''f'''w_x^5w_{2x} + \\
&60(f''')^2w_x^6u_0 + 60(f''')^2f'w_x^4
\end{aligned}
$$

$$
\begin{aligned}
&105u_{2x}u_x^2 = 105(u_0)_x^2(u_0)_{2x} + 105f'w_{3x}(u_0)_x^2 + \\
&210f'w_{2x}(u_0)_x(u_0)_{2x} + 210(f')^2w_{2x}w_{3x}(u_0)_x + \\
&105(f')^2w_{2x}^2(u_0)_{2x} + 105(f')^3w_{2x}^2w_{3x} + \\
&315f''w_xw_{2x}(u_0)_x^2 + 210f''w_x^2(u_0)_x(u_0)_{2x} + \\
&630f'f''w_xw_{2x}^2(u_0)_x + 210f'f''w_x^2w_{3x}(u_0)_x + \\
&210f'f''w_x^2w_{2x}(u_0)_{2x} + 315(f')^2f''w_xw_{2x}^3 + \\
&210(f')^2f''w_x^2w_{2x}w_{3x} + 630(f'')^2w_x^3w_{2x}(u_0)_x + \\
&105(f'')^2w_x^4(u_0)_{2x} + 630f'(f'')^2w_x^3w_{2x}^2 + \\
&105f'(f'')^2w_x^4w_{3x} + 315(f'')^3w_x^5w_{2x} + \\
&105f'''w_x^3(u_0)_x^2 + 210f'f'''w_x^3w_{2x}(u_0)_x + \\
&105(f')^2f'''w_x^3w_{2x}^2 + 210f''f'''w_x^5(u_0)_x + \\
&210f'f''f'''w_x^5w_{2x} + 105(f'')^2f'''w_x^7
\end{aligned}
$$

$$
\begin{aligned}
&20u^3u_{3x} = 20f^{(4)}w_x^4u_0^3 + 20u_0^3(u_0)_{3x} + \\
&60f^{(4)}f'w_x^5u_0^2 + 20f'w_{4x}u_0^3 + 60f'w_xu_0^2(u_0)_{3x} + \\
&60f^{(4)}(f')^2w_x^6u_0 + 60(f')^2w_xw_{4x}u_0^2 + \\
&60(f')^2w_x^2u_0(u_0)_{3x} + 20f^{(4)}(f')^3w_x^7 + \\
&60(f')^3w_x^2w_{4x}u_0 + 20(f')^3w_x^3(u_0)_{3x} + \\
&20(f')^4w_x^3w_{4x} + 60f''w_{2x}^2u_0^3 + 80f''w_xw_{3x}u_0^3 + \\
&180f'f''w_xw_{2x}^2u_0^2 + 240f'f''w_x^2w_{3x}u_0^2 + \\
&180(f')^2f''w_x^2w_{2x}^2u_0 + 240(f')^2f''w_x^3w_{3x}u_0 + \\
&60(f')^3f''w_x^3w_{2x}^2 + 80(f')^3f''w_x^4w_{3x} + \\
&120f'''w_x^2w_{2x}u_0^3 + 360f'f'''w_x^3w_{2x}u_0^2 + \\
&360(f')^2f'''w_x^4w_{2x}u_0 + 120(f')^3f'''w_x^5w_{2x}
\end{aligned}
$$

$$
\begin{aligned}
&150u^2u_xu_{2x} = \\
&150u_0^2(u_0)_x(u_0)_{2x} + 150f'w_{3x}u_0^2(u_0)_x + \\
&150f'w_{2x}u_0^2(u_0)_{2x} + 300f'w_xu_0(u_0)_x(u_0)_{2x} + \\
&150(f')^2w_{2x}w_{3x}u_0^2 + 300(f')^2w_xw_{3x}u_0(u_0)_x + \\
&300(f')^2w_xw_{2x}u_0(u_0)_{2x} + 150(f')^2w_x^2(u_0)_x(u_0)_{2x} + \\
&300(f')^3w_xw_{2x}w_{3x}u_0 + 150(f')^3w_x^2w_{3x}(u_0)_x + \\
&150(f')^3w_x^2w_{2x}(u_0)_{2x} + 150(f')^4w_x^2w_{2x}w_{3x} + \\
&450f''w_xw_{2x}u_0^2(u_0)_x + 150f''w_x^2u_0^2(u_0)_{2x} + \\
&450f'f''w_xw_{2x}^2u_0^2 + 150f'f''w_x^2w_{3x}u_0^2 + \\
&900f'f''w_x^2w_{2x}u_0(u_0)_x + 300f'f''w_x^3u_0(u_0)_{2x} + \\
&900(f')^2f''w_x^2w_{2x}^2u_0 + 300(f')^2f''w_x^3w_{3x}u_0 + \\
&450(f')^2f''w_x^3w_{2x}(u_0)_x + 150(f')^2f''w_x^4(u_0)_{2x} + \\
&450(f')^3f''w_x^3w_{2x}^2 + 150(f')^3f''w_x^4w_{3x} + \\
&450(f'')^2w_x^3w_{2x}u_0^2 + 900f'(f'')^2w_x^4w_{2x}u_0 + \\
&450(f')^2(f'')^2w_x^5w_{2x} + 150f'''w_x^3u_0^2(u_0)_x + \\
&150f'f'''w_x^3w_{2x}u_0^2 + 300f'f'''w_x^4u_0(u_0)_x + \\
&300(f')^2f'''w_x^4w_{2x}u_0 + 150(f')^2f'''w_x^5(u_0)_x + \\
&150(f')^3f'''w_x^5w_{2x} + 150f''f'''w_x^5u_0^2 + \\
&300f'f''f'''w_x^6u_0 + 150(f')^2f''f'''w_x^7
\end{aligned}
$$

$$
\begin{aligned}
&90uu_x^3 = 90u_0(u_0)_x^3 + 270f'u_0(u_0)_x^2w_{2x} + \\
&90f'w_x(u_0)_x^3 + 270(f')^2w_{2x}^2u_0(u_0)_x + \\
&270(f')^2w_xw_{2x}(u_0)_x^2 + 90(f')^3w_{2x}^3u_0 + \\
&270(f')^3w_xw_{2x}^2(u_0)_x + 90(f')^4w_xw_{2x}^3 + \\
&270f''w_x^2u_0(u_0)_x^2 + 540f'f''w_x^2w_{2x}u_0(u_0)_x + \\
&270f'f''w_x^3(u_0)_x^2 + 270(f')^2f''w_x^2w_{2x}^2u_0 + \\
&540(f')^2f''w_x^3w_{2x}(u_0)_x + 270(f')^3f''w_x^3w_{2x}^2 + \\
&270(f'')^2w_x^4u_0(u_0)_x + 270f'(f'')^2w_x^4w_{2x}u_0 + \\
&270f'(f'')^2w_x^5(u_0)_x + 270(f')^2(f'')^2w_x^5w_{2x} + \\
&90(f'')^3w_x^6u_0 + 90f'(f'')^3w_x^7
\end{aligned}
$$

$$
\begin{aligned}
&15u^4u_{2x} = \\
&15u_0^4(u_0)_{2x} + 15f'u_0^4w_{3x} + 60f'w_xu_0^3(u_0)_{2x} + \\
&60(f')^2w_xw_{3x}u_0^3 + 90(f')^2w_x^2u_0^2(u_0)_{2x} + \\
&90(f')^3w_x^2w_{3x}u_0^2 + 60(f')^3w_x^3u_0(u_0)_{2x} + \\
&60(f')^4w_x^3w_{3x}u_0 + 15(f')^4w_x^4(u_0)_{2x} + \\
&15(f')^5w_x^4w_{3x} + 45f''w_xw_{2x}u_0^4 + \\
&180f'f''w_x^2w_{2x}u_0^3 + 270(f')^2f''w_x^3w_{2x}u_0^2 + \\
&180(f')^3f''w_x^4w_{2x}u_0 + 45(f')^4f''w_x^5w_{2x} + \\
&15f'''w_x^3u_0^4 + 60f'f'''w_x^4u_0^3 + 90(f')^2f'''w_x^5u_0^2 + \\
&60(f')^3f'''w_x^6u_0 + 15(f')^4f'''w_x^7
\end{aligned}
$$

$$
\begin{aligned}
&60u^3u_x^2 = 60u_0^3(u_0)_x^2 + 120f'w_{2x}u_0^3(u_0)_x + \\
&180f'w_xu_0^2(u_0)_x^2 + 60(f')^2w_{2x}^2u_0^3 + \\
&360(f')^2w_xw_{2x}u_0^2(u_0)_x + 180(f')^2w_x^2u_0(u_0)_x^2 + \\
&180(f')^3w_xw_{2x}^2u_0^2 + 360(f')^3w_x^2w_{2x}u_0(u_0)_x +
\end{aligned}
$$

Celal Bayar University Journal of Science
Volume 16, Issue 2, 2020, p 229-236
Doi: 10.18466/cbayarfbe.614476

$$60(f')^3w_x^3(u_0)_x^2+180(f')^4w_x^2w_{2x}^2u_0+120(f')^4w_x^3w_{2x}(u_0)_x+60(f')^5w_x^3w_{2x}^2+120f''w_x^2u_0^3(u_0)_x+120f'f''w_x^2w_{2x}u_0^3+360f'f''w_x^3u_0^2(u_0)_x+360(f')^2f''w_x^3w_{2x}u_0^2+360(f')^2f''w_x^4u_0(u_0)_x+360(f')^3f''w_x^4w_{2x}u_0+120(f')^3f''w_x^5(u_0)_x+120(f')^4f''w_x^5w_{2x}+60(f'')^2w_x^4u_0^3+180f'(f'')^2w_x^5u_0^2+180(f')^2(f'')^2w_x^6u_0+60(f')^3(f'')^2w_x^7$$

$$6u^5u_x=6u_0^5(u_0)_x+6f'w_{2x}u_0^5+30f'w_xu_0^4(u_0)_x+30(f')^2w_xw_{2x}u_0^4+60(f')^2w_x^2u_0^3(u_0)_x+60(f')^3w_x^2w_{2x}u_0^3+60(f')^3w_x^3u_0^2(u_0)_x+60(f')^4w_x^3w_{2x}u_0^2+30(f')^4w_x^4u_0(u_0)_x+30(f')^5w_x^4w_{2x}u_0+6(f')^5w_x^5(u_0)_x+6(f')^6w_x^5w_{2x}+6f''w_x^2u_0^5+30f'f''w_x^3u_0^4+60(f')^2f''w_x^4u_0^3+60(f')^3f''w_x^5u_0^2+30(f')^4f''w_x^6u_0+6(f')^5f''w_x^7$$

and

$$w_x^7\big[f^{(7)}+21f^{(5)}f''+35f^{(4)}f'''+6f^{(6)}f'+15f^{(5)}(f')^2+90f^{(4)}f'f''+60(f''')^2f'+105(f'')^2f'''+20f^{(4)}(f')^3+150(f')^2f''f'''+90f'(f'')^3+15(f')^4f'''+60(f')^3(f'')^2+6(f')^5f''\big]$$
$$+\big(21f^{(6)}w_x^5w_{2x}+111f^{(5)}f'w_x^5w_{2x}+315f^{(4)}f''w_x^5w_{2x}+210(f''')^2w_x^5w_{2x}+6f^{(6)}w_x^6u_0+30f^{(5)}f'w_x^6u_0+240f^{(4)}(f')^2w_x^5w_{2x}+90f^{(4)}f''w_x^6u_0+1110f'f''f'''w_x^5w_{2x}+60(f''')^2w_x^6u_0+315(f'')^3w_x^5w_{2x}+60f^{(4)}(f')^2w_x^6u_0+270(f')^3f'''w_x^5w_{2x}+720(f')^2(f'')^2w_x^5w_{2x}+300f'f''f'''w_x^6u_0+90(f'')^3w_x^6u_0+165(f')^4f''w_x^5w_{2x}+60(f')^3f'''w_x^6u_0+180(f')^2(f'')^2w_x^6u_0+6(f')^6w_x^5w_{2x}+30(f')^4f''w_x^6u_0\big)$$
$$+\big(105f^{(5)}w_x^3w_{2x}^2+35f^{(5)}w_x^4w_{3x}+21f^{(5)}w_x^5(u_0)_x+480f^{(4)}f'w_x^3w_{2x}^2+1050f''f'''w_x^3w_{2x}^2+350f''f'''w_x^4w_{3x}+155f^{(4)}f'w_x^4w_{3x}+90f^{(5)}w_x^4w_{2x}u_0+15f^{(5)}w_x^5u_0^2+390f^{(4)}f'w_x^4w_{2x}u_0+270(f')^2f'''w_x^4w_{3x}+90f^{(4)}f'w_x^5(u_0)_x+1440f'(f'')^2w_x^3w_{2x}^2+465f'(f'')^2w_x^4w_{3x}+870(f')^2f'''w_x^3w_{2x}^2+900f''f'''w_x^4w_{2x}u_0+210f''f'''w_x^5(u_0)_x+60f^{(4)}f'w_x^5u_0^2+780(f')^3f''w_x^3w_{2x}^2+230(f')^3f''w_x^4w_{3x}+660(f')^2f'''w_x^4w_{2x}u_0+1170f'(f'')^2w_x^4w_{2x}u_0+150(f')^2f'''w_x^5(u_0)_x+150f''f'''w_x^5u_0^2+270f'(f'')^2w_x^5(u_0)_x+15(f')^5w_x^4w_{3x}+540(f')^3f''w_x^4w_{2x}u_0+90(f')^2f'''w_x^5u_0^2+60(f')^5w_x^3w_{2x}^2+120(f')^3f''w_x^5(u_0)_x+180f'(f'')^2w_x^5u_0^2+30(f')^5w_x^4w_{2x}u_0+6(f')^5w_x^5(u_0)_x+60(f')^3f''w_x^5u_0^2\big)$$
$$+\big(105f^{(4)}w_xw_{2x}^3+210f^{(4)}w_x^2w_{2x}w_{3x}+35f^{(4)}w_x^3w_{4x}+210f^{(4)}w_x^3w_{2x}(u_0)_x+105(f'')^2w_x^3w_{4x}+405f'f'''w_xw_{2x}^3+35f^{(4)}w_x^4(u_0)_{2x}+315(f'')^2w_xw_{2x}^3+630(f'')^2w_x^2w_{2x}w_{3x}+125f'f'''w_x^3w_{4x}+270f^{(4)}w_x^2w_{2x}^2u_0+120f^{(4)}w_x^3w_{3x}u_0+780f'f'''w_x^2w_{2x}w_{3x}+150f^{(4)}w_x^3w_{2x}u_0^2+990f'f'''w_x^2w_{2x}^2u_0+420f'f'''w_x^3w_{3x}u_0+585(f')^2f''w_xw_{2x}^3+1080(f')^2f''w_x^2w_{2x}w_{3x}+165(f')^2f''w_x^3w_{4x}+810(f'')^2w_x^2w_{2x}^2u_0+360(f'')^2w_x^3w_{3x}u_0+750f'f'''w_x^3w_{2x}(u_0)_x+120f'f'''w_x^4(u_0)_{2x}+630(f'')^2w_x^3w_{2x}(u_0)_x+105(f'')^2w_x^4(u_0)_{2x}+20f^{(4)}w_x^4u_0^3+20(f')^4w_x^3w_{4x}+150(f')^4w_x^2w_{2x}w_{3x}+540(f')^2f''w_x^3w_{3x}u_0+990(f')^2f''w_x^3w_{2x}(u_0)_x+150(f')^2f''w_x^4(u_0)_{2x}+450(f'')^2w_x^3w_{2x}u_0^2+510f'f'''w_x^3w_{2x}u_0^2+300f'f'''w_x^4u_0(u_0)_x+90(f')^4w_xw_{2x}^3+1350(f')^2f''w_x^2w_{2x}^2u_0+270(f'')^2w_x^4u_0(u_0)_x+60(f')^4w_x^3w_{3x}u_0+15(f')^4w_x^4(u_0)_{2x}+60f'f'''w_x^4u_0^3+180(f')^4w_x^2w_{2x}^2u_0+120(f')^4w_x^3w_{2x}(u_0)_x+630(f')^2f''w_x^3w_{2x}u_0^2+360(f')^2f''w_x^4u_0(u_0)_x+60(f'')^2w_x^4u_0^3+60(f')^4w_x^3w_{2x}u_0^2+30(f')^4w_x^4u_0(u_0)_x+60(f')^2f''w_x^4u_0^3\big)$$
$$+\big(105f'''w_{2x}^2w_{3x}+70f'''w_xw_{3x}^2+105f'''w_xw_{2x}w_{4x}+21f'''w_x^2w_{5x}+315f'f''w_{2x}^2w_{3x}+300f'f''w_xw_{2x}w_{4x}+57f'f''w_x^2w_{5x}+315f'''w_xw_{2x}^2(u_0)_x+210f'''w_x^2w_{3x}(u_0)_x+200f'f''w_xw_{3x}^2+35f'''w_x^3(u_0)_{3x}+210f'''w_x^2w_{2x}(u_0)_{2x}+90f'''w_{2x}^3u_0+360f'''w_xw_{2x}w_{3x}u_0+90f'''w_x^2w_{4x}u_0+15(f')^3w_x^2w_{5x}+1020f'f''w_xw_{2x}w_{3x}u_0+240f'f''w_x^2w_{4x}u_0+225f'''w_xw_{2x}^2u_0^2+150f'''w_x^2w_{3x}u_0^2+270f'f''w_{2x}^3u_0+900f'f''w_xw_{2x}^2(u_0)_x+570f'f''w_x^2w_{3x}(u_0)_x+90f'f''w_x^3(u_0)_{3x}+540f'''w_x^2w_{2x}u_0(u_0)_x+90(f')^3w_xw_{2x}w_{4x}+60(f')^3w_xw_{3x}^2+570f'f''w_x^2w_{2x}(u_0)_{2x}+120f'''w_x^3u_0(u_0)_{2x}+105(f')^3w_{2x}^2w_{3x}+105f'''w_x^3(u_0)_x^2+60(f')^3w_x^2w_{4x}u_0+20(f')^3w_x^3(u_0)_{3x}+630f'f''w_xw_{2x}^2u_0^2+390f'f''w_x^2w_{3x}u_0^2+120f'''w_x^2w_{2x}u_0^3+300(f')^3w_xw_{2x}w_{3x}u_0+150(f')^3w_x^2w_{3x}(u_0)_x+150(f')^3w_x^2w_{2x}(u_0)_{2x}+1440f'f''w_x^2w_{2x}u_0(u_0)_x+300f'f''w_x^3u_0(u_0)_{2x}+150f'''w_x^3u_0^2(u_0)_x+90(f')^3w_{2x}^3u_0+270(f')^3w_xw_{2x}^2(u_0)_x+270f'f''w_x^3(u_0)_x^2+90(f')^3w_x^2w_{3x}u_0^2+60(f')^3w_x^3u_0(u_0)_{2x}+300f'f''w_x^2w_{2x}u_0^3+15f'''w_x^3u_0^4+180(f')^3w_xw_{2x}^2u_0^2+360(f')^3w_x^2w_{2x}u_0(u_0)_x+60(f')^3w_x^3(u_0)_x^2+360f'f''w_x^3u_0^2(u_0)_x+60(f')^3w_x^2w_{2x}u_0^3+60(f')^3w_x^3u_0^2(u_0)_x+30f'f''w_x^3u_0^4\big)$$
$$+\big(f''w_tw_x+35f''w_{3x}w_{4x}+21f''w_{2x}w_{5x}+7f''w_xw_{6x}+21(f')^2w_{2x}w_{5x}+210f''w_{2x}w_{3x}(u_0)_x+105f''w_xw_{4x}(u_0)_x+21f''w_x^2(u_0)_{4x}+35(f')^2w_{3x}w_{4x}+105f''w_xw_{2x}(u_0)_{3x}+105f''w_{2x}^2(u_0)_{2x}+140f''w_xw_{3x}(u_0)_{2x}+60f''w_{3x}^2u_0+90f''w_{2x}w_{4x}u_0+36f''w_xw_{5x}u_0+6(f')^2w_xw_{6x}+30(f')^2w_xw_{5x}u_0+15(f')^2w_x^2(u_0)_{4x}+150f''w_{2x}w_{3x}u_0^2+75f''w_xw_{4x}u_0^2+90(f')^2w_{2x}w_{4x}u_0+$$

Celal Bayar University Journal of Science
Volume 16, Issue 2, 2020, p 229-236
Doi: 10.18466/cbayarfbe.614476

$90(f')^2 w_x w_{4x}(u_0)_x + 90(f')^2 w_x w_{2x}(u_0)_{3x} + 270f'' w_{2x}^2 u_0 (u_0)_x + 360f'' w_x w_{3x} u_0 (u_0)_x + 90f'' w_x^2 u_0 (u_0)_{3x} + 60(f')^2 w_{3x}^2 u_0 + 120(f')^2 w_x w_{3x}(u_0)_{2x} + 360f'' w_x w_{2x} u_0 (u_0)_{2x} + 210(f')^2 w_{2x} w_{3x}(u_0)_x + 105(f')^2 w_{2x}^2 (u_0)_{2x} + 315f'' w_x w_{2x}(u_0)_x^2 + 210f'' w_x^2 (u_0)_x (u_0)_{2x} + 60(f')^2 w_x w_{4x} u_0^2 + 60(f')^2 w_x^2 u_0 (u_0)_{3x} + 60f'' w_{2x}^2 u_0^3 + 80f'' w_x w_{3x} u_0^3 + 150(f')^2 w_{2x} w_{3x} u_0^2 + 300(f')^2 w_x w_{3x} u_0 (u_0)_x + 300(f')^2 w_x w_{2x} u_0 (u_0)_{2x} + 150(f')^2 w_x^2 (u_0)_x (u_0)_{2x} + 450f'' w_x w_{2x} u_0^2 (u_0)_x + 150f'' w_x^2 u_0^2 (u_0)_{2x} + 270(f')^2 w_{2x}^2 u_0 (u_0)_x + 270(f')^2 w_x w_{2x}(u_0)_x^2 + 270f'' w_x^2 u_0 (u_0)_x^2 + 60(f')^2 w_x w_{3x} u_0^3 + 90(f')^2 w_x^2 u_0^2 (u_0)_{2x} + 45f'' w_x w_{2x} u_0^4 + 60(f')^2 w_{2x}^2 u_0^3 + 360(f')^2 w_x w_{2x} u_0^2 (u_0)_x + 180(f')^2 w_x^2 u_0 (u_0)_x^2 + 120f'' w_x^2 u_0^3 (u_0)_x + 30(f')^2 w_x w_{2x} u_0^4 + 60(f')^2 w_x^2 u_0^3 (u_0)_x + 6f'' w_x^2 u_0^5) + (f' w_{xt} + f' w_{7x} + 21f' w_{5x}(u_0)_x + 21f' w_{2x}(u_0)_{4x} + 35f' w_{3x}(u_0)_{3x} + 35f' w_{4x}(u_0)_{2x} + 6f' w_{6x} u_0 + 6f' w_x (u_0)_{5x} + 15f' w_{5x} u_0^2 + 30f' w_x u_0 (u_0)_{4x} + 90f' w_{4x} u_0 (u_0)_x + 90f' w_{2x} u_0 (u_0)_{3x} + 90f' w_x (u_0)_x (u_0)_{3x} + 120f' w_{3x} u_0 (u_0)_{2x} + 60f' w_x (u_0)_{2x}^2 + 105f' w_{3x}(u_0)_x^2 + 210f' w_{2x}(u_0)_x (u_0)_{2x} + 20f' w_{4x} u_0^3 + 60f' w_x u_0^2 (u_0)_{3x} + 150f' w_{3x} u_0^2 (u_0)_x + 150f' w_{2x} u_0^2 (u_0)_{2x} + 300f' w_x u_0 (u_0)_x (u_0)_{2x} + 270f' w_{2x} u_0 (u_0)_x^2 + 90f' w_x (u_0)_x^3 + 15f' w_{3x} u_0^4 + 60f' w_x u_0^3 (u_0)_{2x} + 120f' w_{2x} u_0^3 (u_0)_x + 180f' w_x u_0^2 (u_0)_x^2 + 6f' w_{2x} u_0^5 + 30f' w_x u_0^4 (u_0)_x).$ **(3.18)**

Setting the coefficients of w_x^7 in (3.18) to zero respectively, we obtain a set of ordinary differential equations

$$f^{(7)} + 21f^{(5)}f'' + 35f^{(4)}f''' + 6f^{(6)}f' + 15f^{(5)}(f')^2 + 90f^{(4)}f'f'' + 60(f''')^2 f' + 105(f'')^2 f''' + 20f^{(4)}(f')^3 + 150(f')^2 f'' f''' + 90f'(f'')^3 + 15(f')^4 f''' + 60(f')^3(f'')^2 + 6(f')^5 f'' = 0,$$

which have solutions

$$f = lnw, \quad \textbf{(3.19)}$$

there by from (21) it holds that

$$(f')^2 = -f'',\ (f')^3 = \tfrac{1}{2}f''',\ f'f'' = -\tfrac{1}{2}f''',\ (f')^4 = (f'')^2 = -\tfrac{1}{6}f^{(4)},$$
$$f'f''' = -\tfrac{1}{3}f^{(4)},\ (f')^2 f'' = \tfrac{1}{6}f^{(4)},\ f^{(4)}f' = -\tfrac{1}{4}f^{(5)},\ f''f''' = -\tfrac{1}{12}f^{(5)},$$

$$(f')^2 f''' = \tfrac{1}{12}f^{(5)},$$
$$(f')^5 = f'(f'')^2 = \tfrac{1}{24}f^{(5)},\ (f')^3 f'' = -\tfrac{1}{24}f^{(5)},\ f^{(5)}f' = -\tfrac{1}{5}f^{(6)},$$
$$f^{(4)}f'' = -\tfrac{1}{20}f^{(6)}, f^{(4)}(f')^2 = \tfrac{1}{20}f^{(6)}, f'f''f''' = \tfrac{1}{60}f^{(6)}, (f''')^2 = -\tfrac{1}{30}f^{(6)},$$

$$(f'')^3 = (f')^4 f'' = \tfrac{1}{120}f^{(6)}, (f')^3 f''' = -\tfrac{1}{60}f^{(6)},$$
$$(f')^2(f'')^2 = (f')^6 = -\tfrac{1}{120}f^{(6)}. \quad \textbf{(3.20)}$$

By using Eqs. (3.18) and (3.20) can be written as the sum of some terms with f' and f'' setting their coefficients to zero will lead to

$$w_x(w_t + w_{6x} + 15w_{4x}(u_0)_x + 6w_x(u_0)_{4x} + 15w_{2x}(u_0)_{3x} + 20w_{3x}(u_0)_{2x} + 6w_{5x}u_0 + 15w_{4x}u_0^2 + 60w_{3x}u_0(u_0)_x + 30w_x u_0(u_0)_{3x} + 60w_{2x}u_0(u_0)_{2x} + 45w_{2x}(u_0)_x^2 + 60w_x(u_0)_x(u_0)_{2x} + 20w_{3x}u_0^3 + 90w_{2x}u_0^2(u_0)_x + 60w_x u_0^2(u_0)_{2x} + 90w_x u_0(u_0)_x^2 + 15w_{2x}u_0^4 + 60w_x u_0^3(u_0)_x + 6w_x u_0^5) + \frac{\partial}{\partial x}(w_t + w_{6x} + 15w_{4x}(u_0)_x + 6w_x(u_0)_{4x} + 15w_{2x}(u_0)_{3x} + 20w_{3x}(u_0)_{2x} + 6w_{5x}u_0 + 15w_{4x}u_0^2 + 60w_{3x}u_0(u_0)_x + 30w_x u_0(u_0)_{3x} + 60w_{2x}u_0(u_0)_{2x} + 45w_{2x}(u_0)_x^2 + 60w_x(u_0)_x(u_0)_{2x} + 20w_{3x}u_0^3 + 90w_{2x}u_0^2(u_0)_x + 60w_x u_0^2(u_0)_{2x} + 90w_x u_0(u_0)_x^2 + 15w_{2x}u_0^4 + 60w_x u_0^3(u_0)_x + 6w_x u_0^5) = 0.$$
(3.21)

Above equations are satisfied provided that

$$w_t + w_{6x} + 15w_{4x}(u_0)_x + 6w_x(u_0)_{4x} + 15w_{2x}(u_0)_{3x} + 20w_{3x}(u_0)_{2x} + 6w_{5x}u_0 + 15w_{4x}u_0^2 + 60w_{3x}u_0(u_0)_x + 30w_x u_0(u_0)_{3x} + 60w_{2x}u_0(u_0)_{2x} + 45w_{2x}(u_0)_x^2 + 60w_x(u_0)_x(u_0)_{2x} + 20w_{3x}u_0^3 + 90w_{2x}u_0^2(u_0)_x + 60w_x u_0^2(u_0)_{2x} + 90w_x u_0(u_0)_x^2 + 15w_{2x}u_0^4 + 60w_x u_0^3(u_0)_x + 6w_x u_0^5 = 0. \quad \textbf{(3.22)}$$

From (3.17) and (3.19), we obtain Bӓcklund transformation of Eq.(3.16)

$$u = \frac{\partial}{\partial x} lnw + u_0, \quad \textbf{(3.23)}$$

where w satisfying (3.22). We take initial solutions of Eq.(3.16) as $u_0 = 0$, then (3.22) reduce to

$$w_t + w_{6x} = 0, \quad \textbf{(3.24)}$$

$$u = \frac{\partial}{\partial x} lnw. \quad \textbf{(3.25)}$$

Specially, we take a solution of (3.24)

$$w = 1 + e^{c(x - c^5 t)}. \quad \textbf{(3.26)}$$

Then, solutions of Eq.(3.16) can be written by using (3.25) as following

$$u(x,t) = \frac{ce^{c(x-c^5t)}}{1+e^{c(x-c^5t)}}, \quad \textbf{(3.27)}$$

or

$$u(x,t) = \frac{c}{2}\left(1 + \mathrm{Tanh}\left[\frac{c}{2}(x - c^5 t)\right]\right). \tag{3.28}$$

where c is arbitrary constant.

4. Results and Discussion

We discuss the results to Eq. (3.1) and Eq. (3.16) obtained by using the present technique in the literature and the reported results in this study. J. Liu et al. [43] obtained the exact solutions of complexly coupled KdV equations by using Jacobi elliptic function method. I.E. Inan and D. Kaya [38] obtained exact solutions of complexly coupled KdV equations by using Generalized tanh function method. A.M. Wazwaz [35] obtained multiple kink solutions and multiple singular kink solutions of Burgers hierarchy equations by using Hirota bilinear method. For Eq (3.1), when we compare our result with the results reported in J. Liu et al. [43] and I.E. Inan and D. Kaya [38], we found the solution is similar to the hyperbolic solution found by J. Liu et al. [43] with I.E. Inan and D. Kaya [38]. For Eq (18), when we compare our result with the result reported in A.M. Wazwaz [35], we found the solution is same to the hyperbolic solution found by A.M. Wazwaz [35]. The Auto- B$\ddot{a}$cklund transformation used in this article is a powerful method for finding traveling wave solutions of nonlinear partial differential equations.

5. Conclusion

We used the Auto-B$\ddot{a}$cklund transformation for finding the travelling wave solutions of the complexly coupled KdV equations and the sixth order equation of the Burgers hierarchy.. The solutions found as a result of the application of the method are hyperbolic function and exponential function solutions. The accuracy of these solutions was seen by using Mathematica 11.2 computer program. Many nonlinear partial differential equations and system of equations can be solved using this method.

Author's Contributions

All authors contributed equally to this manuscript and all authors reviewed the final manuscript.

Ethics

There are no ethical issues after the publication of this manuscript.

References

[1]. Shang, Y. 2007. Backlund transformation, Lax pairs and explicit exact solutions for the shallow water waves equation. *Applied Mathematics and Computation*;, 187: 1286-1297.

[2]. Bock, TL, Kruskal, MD. 1979. A two-parameter Miura transformation of the Benjamin-Ono equation. *Physics Letters A*; 74: 173-176.

[3]. Abourabia, A, El Horbaty MM. 2006. On solitary wave solutions for the two-dimensional nonlinear modified Kortweg-de Vries-Burger equation. *Chaos Solitons Fractals*; 29: 354-364.

[4]. Malfliet, W. 1992. Solitary wave solutions of nonlinear wave equations. *American Journal of Physics*; 60: 650-654.

[5]. Chuntao, Y. 1996. A simple transformation for nonlinear waves. *Physics Letters A*; 224: 77-84.

[6]. Cariello, F, Tabor, M. 1989. Painleve expansions for nonintegrable evolution equations. *Physica D*; 39: 77-94.

[7]. Fan, E. 2000. Two new application of the homogeneous balance method. *Physics Letters A*; 265: 353-357.

[8]. Clarkson, PA. 1989. New similarity solutions for the modified boussinesq equation, *Journal of Physics A: Mathematical and General*; 22: 2355-2367.

[9]. Fan, E. 2000. Extended tanh-function method and its applications to nonlinear equations, *Physics Letters A*; 277: 212-218.

[10]. Elwakil, S A, El-labany, SK, Zahran, MA,d Sabry R. 2002. Modified extended tanh-function method for solving nonlinear partial differential equations. *Physics Letters A*; 299: 179-188.

[11]. Chen, H, Zhang, H. 2004. New multiple soliton solutions to the general Burgers-Fisher equation and the Kuramoto-Sivashinsky equation. *Chaos, Solitons and Fractals*; 19: 71-76.

[12]. Wazwaz, AM. 2008. Analytic study on the one and two spatial dimensional potential KdV equations. *Chaos Solitons and Fractals;* 36: 175–181.

[13]. Kudryashov, N.A. 2019. The Painlevé approach for finding solitary wave solutions of nonlinear nonintegrable differential equations. *Optik*; 183: 642–649.

[14]. Chen, H T, Hong-Qing, Z. 2004. New double periodic and multiple soliton solutions of the generalized (2+1)-dimensional Boussinesq equation. *Chaos, Solitons and Fractals;* 20: 765-769.

[15]. Wazwaz, AM. 2018. Two-mode Sharma-Tasso-Olver equation and two-mode fourth-order Burgers equation: Multiplekink solutions. *Alexandria Engineering Journal*;57: 1971–1976.

[16]. Chen, Y, Yan, Z. 2006. The Weierstrass elliptic function expansion method and its applications in nonlinear wave equations. *Chaos Solitons and Fractals*; 29: 948-964.

[17]. Wazwaz, AM. 2010. Burgers hierarchy: Multiple kink solutions and multiple singular kink solutions. *Journal of the Franklin Institute*; 347: 618–626.

[18]. Guo, S, Zhou, Y. 2010. The extended $\left(\frac{G'}{G}\right)$ -expansion method and its applications to the Whitham-Broer-Kaup-like equations and coupled Hirota-Satsuma KdV equations. *Applied Mathematics and* Computation; 215: 3214-3221.

[19]. Inan, IE, Ugurlu, Y, Bulut H. 2016. Auto-B$\ddot{a}$cklund transformation for some nonlinear partial differential equations. *Optik*;127: 10780-10785.

[20]. Manafian J, Lakestain, M. 2016. Application of $\tan\left(\frac{\Phi(\varphi)}{2}\right)$-expansion method for solving the Biswas-Milovic equation for Kerr law nonlinearity. *Optik*; 127: 2040-2054.

[22]. Zhou, Q, Ekici, M, Sonmezoglu, A, Mirzazadeh, M. 2016. Optical solitons with Biswas–Milovic equation by extended $\left(\frac{G'}{G}\right)$-expansion method. *Optik*; 127: 6277–6290.

[23]. Ebadi, G, Biswas, A. 2010. Application of the $\left(\frac{G'}{G}\right)$-expansion method for nonlinear diffusion equations with nonlinear source. *Journal of the Franklin Institute*; 347: 1391–1398.

[24]. Kudryashov, NA. 2018. Exact solutions and integrability of the Duffing–Van der Pol equation, Regul. *Chaotic Dyn;* 23 (4): 471–479.

[25]. Inan, IE, Kaya, D. 2007. Exact solutions of some nonlinear partial differential equations. *Physica A*; 381: 104-115.

[26]. Biswas, A, Ekici, M, Sonmezoglu, A, Belic, M.R. 2019. Highly dispersive optical solitons with Kerr law nonlinearity by F–expansion. *Optik* ;181: 1028–1038

[21]. Manafian, J, Aghdaei, MF, Khalilian M, Jeddi, R.S. 2017. Application of the generalized $\left(\frac{G'}{G}\right)$ -expansion method for nonlinear PDEs to obtaining soliton wave solution. *Optik*; 135: 395–406.

[27]. Biswas, A, Ekici, M, Sonmezoglu, A, Belic, M.R. 2019. Highly dispersive optical solitons with quadratic–cubic law by F-expansion. *Optik* ;182: 930–943.

[28]. Liu, J, Yang, L, Yang, K. 2004. Jacobi elliptic function solutions of some nonlinear PDEs. *Physics Letters A*; 325: 268-275.

[29]. Pandir, Y. 2017. A New Type of Generalized F-Expansion Method and its Application to Sine-Gordon Equation. *Celal Bayar University Journal of Science*; 13: (3) 647-650.

[30]. Yaşar, E, Giresunlu, İB. 2017. Symmetry Reductions, Exact Solutions and Conservation Laws for the Coupled Nonlinear Klein-Gordon System. *Celal Bayar University Journal of Science*; 13: (3) 593-599.